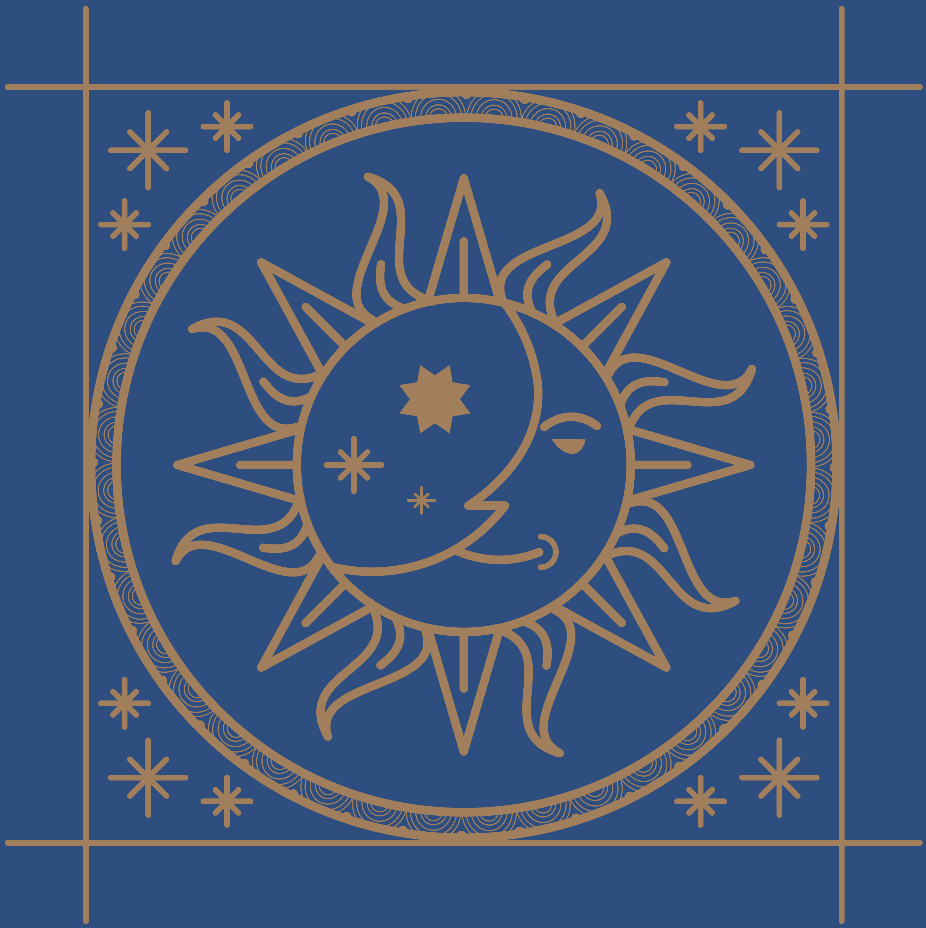


符号: ":=" 定义为 ; "≡" 记作 .

号差 $(-,+,+,+)$

Manifolds and tangent vectors

$\mathscr{F}_M$ 是 M 上所有 C^∞ 映射(函数) $f: M \to \mathbb{R}$ 的集合

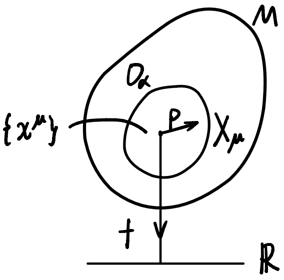
M 上 P 点的矢量 v 是一个线性映射 $v: \mathscr{F}_M \to \mathbb{R}$,满足:

对于 $\forall f, g \in \mathscr{F}_M$, $\alpha, \beta \in \mathbb{R}$,

(1) (线性性) $v(\alpha f + \beta g) = \alpha v(f) + \beta v(f)$,

(2) (莱布尼茨律) $v(fg) = g|_p v(f) + f|_p v(g)$,

(注: 对于 $x \in M$, $fg(x) = f(x)\, g(x)$)



$$X_\mu(f) := \frac{\partial F(x)}{\partial x^\mu}\Big|_p \in \mathbb{R}$$

Claim: ① 构成 vector space

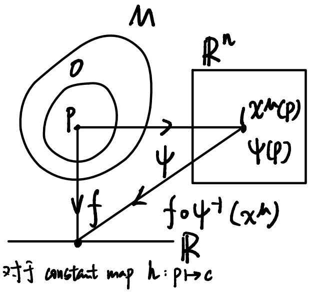
② $\dim V_p = \dim M \equiv n$.

Pf: ① $(v+u)(f) := v(f) + u(f)$

$(\alpha v)(f) := \alpha \cdot v(f)$

$0(f) := 0 \in \mathbb{R}$

② (a) linear independent $X_\mu(f) = \frac{\partial}{\partial x^\mu} f \circ \psi^{-1}\big|_{\psi(p)}$



显然 $X_\mu \in V_p$

$\alpha^\mu X_\mu(f) = 0$ for any f , 不妨令 $f = x^\nu$

$\Rightarrow \alpha^\mu \delta^\nu{}_\mu = 0 \Rightarrow \alpha^\nu = 0$ 所以线性无关

(b) Span V_p

由多元函数的中值定理

for any C^∞ map f, exists C^∞ func. $H_\mu(x^1 \cdots x^n)$

$f(q) = f(p) + (x^\mu(q) - x^\mu(p)) H_\mu(\psi(q))$

且 $H_\mu(\psi(p)) = \frac{\partial f \circ \psi}{\partial x^\mu}\Big|_{\psi(p)} = X_\mu(f)$

$\Rightarrow v(f)|_q = 0 + v(x^\mu)|_q H_\mu(\psi(q)) + (x^\mu(q) - x^\mu(p))\, v(H^\mu \circ \psi)|_q$

当 $q = p$ 时

$v(f)|_p = v(x^\mu)|_p X_\mu(f)$ for $\forall v \in V_p$

$\Rightarrow$ Span V_p □

对于 constant map $h: p \mapsto c$

$v(h^2) = 2c\,v(h) = 2v(ch)$ 而 $h^2(p) = ch(p)$

$\therefore v(h) = 0$

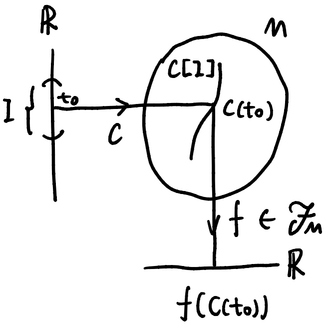
坐标变换:

$\{x^\mu\}$ 变换为 $\{x'^\mu\}$, v 变为 v'

应该有 $v(f) = v'(f)$, 即 $v^\mu X_\mu(f) = v'^\mu X'_\mu(f)$

$$\Rightarrow v^\mu \frac{\partial f(x)}{\partial x^\mu}\Big|_p = v'^\mu \frac{\partial f'(x')}{\partial x'^\mu}\Big|_p \text{ 而 } \frac{\partial f(x)}{\partial x^\mu}\Big|_p = \frac{\partial f'(x')}{\partial x'^\nu}\Big|_p \frac{\partial x'^\nu}{\partial x^\mu}\Big|_p$$

$$\therefore v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu}\Big|_p v^\mu$$



$C: I \to M$ 是 C^r 类 map

称为 C^r 类曲线

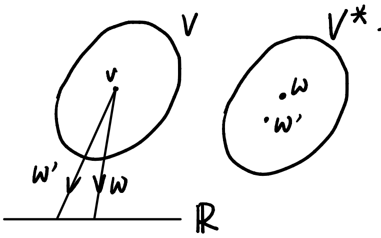
定义切矢量 T, 对 $\forall f \in \mathscr{F}_M$ 作用为

$$T(f) := \frac{\partial (f \circ C)}{\partial t}\Big|_{t_0} \quad (\text{其实也可以用 } \frac{d}{dt} \text{ 表示})$$

按这个定义, $X_\mu \equiv \frac{\partial}{\partial x^\mu}$ 就是曲线 x^μ 的切矢量

那么 $\frac{\partial}{\partial t} = \frac{dx^\mu}{dt} \frac{\partial}{\partial x^\mu}$

Dual Vector



$\to$ dual space

$w: V \to \mathbb{R}$ linear map

w 是一个 dual vector

Claim: ① V^* is a vector space

② $\dim V^* = \dim V$

Pf. ① $(w_1 + w_2)(v) = w_1(v) + w_2(v)$

$(\alpha w)(v) = \alpha \cdot w(v)$

$0(v) = 0 \in \mathbb{R}$

② V 的一组基矢 $e_1 \cdots e_n$

对应 V^* 有一组对偶矢量 $e^{1*} \cdots e^{n*}$

$e^{\mu*}(e_\nu) := \delta^\mu{}_\nu$

(a) $e^{\mu*}$ 线性独立 (只要作用于基矢为 0, 那么, 作用于任一矢量都为 0)

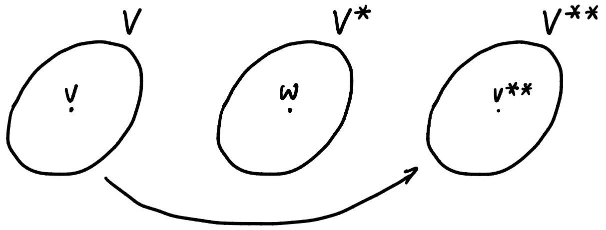
$\alpha_\mu e^{\mu*}(e_\nu) = \alpha_\nu = 0$

$\Rightarrow$ 线性独立

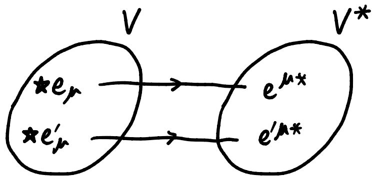
(b) $\forall w \in V^*$, w 可以被 $e^{\mu*}$ 线性表出

令 $w_\mu \equiv w(e_\mu)$, 则 $w = w_\mu e^{*\mu}$

($w_\nu = w_\mu e^{*\mu}(e_\nu) = w_\mu \delta^\mu{}_\nu$, 成立)



$w(v) = v^{**}(w)$



Claim: $e'_\mu = A^\nu{}_\mu e_\nu \Rightarrow e'^{\mu *} = (\tilde{A}^{-1})_\nu{}^\mu e^{\nu *}$

Pf. $(\tilde{A}^{-1})_\nu{}^\mu e^{\nu *}(A^\beta{}_\alpha e_\beta)$

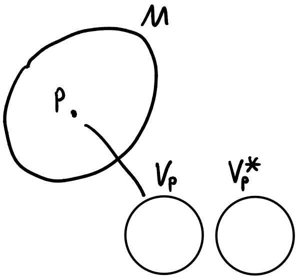
$= (\tilde{A}^{-1})_\nu{}^\mu A^\beta{}_\alpha \, e^{\nu *}(e_\beta)$

$= (A^{-1})^\mu{}_\nu A^\nu{}_\alpha = \delta^\mu{}_\alpha$

$e'^{*\mu}(e'_\alpha) = \delta^\mu{}_\alpha$ （只要作用于基矢是相等的 那么,作用于任一矢量都相等）

∴ 左 = 右 □

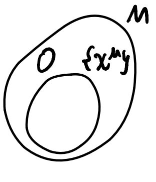
Dual Vector Field on a Manifold



用 $df|_p$ 表示 p上的 dual vector

$df|_p(v) := v(f), \forall v \in V_p, f \in \mathscr{F}_M$

那么 df 就是一个 dual vector field



选定一个 chart, 那么自然在O上有一个 dual vector field dx^μ

$dx^\mu\left(\frac{\partial}{\partial x^\nu}\right) = \delta^\mu{}_\nu$

↓ ↓

dual coordinate vector basis　coordinate vector basis

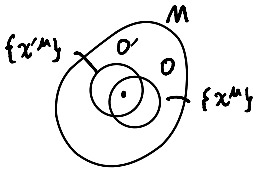
Claim: $df = \frac{\partial f}{\partial x^\mu} dx^\mu = df|_p\left(\frac{\partial}{\partial x^\mu}\right) dx^\mu$

Pf: $df|_p(v) = v(f)|_p = v^\mu \frac{\partial}{\partial x^\mu}(f)|_p = dx^\mu(v)|_p \frac{\partial}{\partial x^\mu}(f)|_p$

↓ ↓

(for ∀ $v \in V_p$)　$v^\mu = dx^\mu\left(v^\nu \frac{\partial}{\partial x^\nu}\right) = v^\nu \delta^\mu{}_\nu$

坐标变换:



$v = v^\mu \frac{\partial}{\partial x^\mu} = v'^\nu \frac{\partial}{\partial x'^\nu}$

$\omega = \omega_\mu dx^\mu = \omega'_\nu dx'^\nu$

∀ $v \in V$

$\omega_\mu dx^\mu(v) = \omega'_\nu dx'^\nu(v)$

$\Rightarrow \omega_\mu dx^\mu\left(v^\alpha \frac{\partial}{\partial x^\alpha}\right) = \omega'_\nu dx'^\nu\left(v'^\beta \frac{\partial}{\partial x'^\beta}\right)$

$\Rightarrow \omega_\mu v^\mu = \omega'_\nu v'^\nu$

而 $v'^\nu = \frac{\partial x'^\nu}{\partial x^\mu}\Big|_p v^\mu$

∴ $\omega_\mu v^\mu = \omega'_\nu \frac{\partial x'^\nu}{\partial x^\mu}\Big|_p v^\mu$

∴ $\omega_\mu = \omega'_\nu \frac{\partial x'^\nu}{\partial x^\mu}$

Tensor Fields on a Manifold

Def. A multiplelinear map

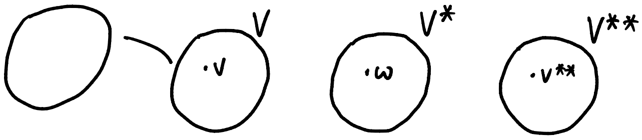
$$T: \underbrace{V^* \times V^* \cdots \times V^*}_{k} \times \underbrace{V \times \cdots \times V}_{l} \to \mathbb{R}$$

is called a tensor of type (k, l) over V

$\omega: V \to \mathbb{R}$ (0,1)

$v \equiv v^{**}: V^* \to \mathbb{R}$ (1,0)

$\mathscr{T}_V(k,l)$



$V = \mathscr{T}_V(1,0)$, $V^* = \mathscr{T}_V(0,1)$

对于一个 type (1,1) 的 tensor $T \in \mathcal{T}_V(1,1)$

$T = T(\cdot\,;\cdot)$ $\quad T(\omega;\cdot) \in V^*$

$\therefore \quad T : V^* \xrightarrow{\text{linear}} V^*$

同样 $T : V \xrightarrow{\text{linear}} V$

Claim: ① $\mathcal{T}_V(k,l)$ is a vector space

② $\dim \mathcal{T}_V(k,l) = n^{k+l}$

Pf: ① $(\alpha T_1 + \beta T_2)(\omega^1 \cdots \omega^k; v_1 \cdots v_l)$

$= \alpha T_1(\omega^1 \cdots \omega^k; v_1 \cdots v_l) + \beta T_2(\omega^1 \cdots \omega^k; v_1 \cdots v_l)$

$O(\omega^1 \cdots \omega^k; v_1 \cdots v_l) = 0 \in \mathbb{R}$

② $\underbrace{e_\mu \otimes \cdots \otimes e_\nu}_{k} \otimes \underbrace{e^{\alpha *} \otimes \cdots \otimes e^{\beta *}}_{l}$ 是 n^{k+l} 个基

(a) 线性无关 显然

(b) $T = T^{\mu \cdots \nu}{}_{\alpha \cdots \beta}\, e_\mu \otimes \cdots \otimes e_\nu \otimes e^{\alpha *} \otimes \cdots \otimes e^{\beta *}$

where $T^{\mu \cdots \nu}{}_{\alpha \cdots \beta} \equiv T(e^{\mu *}, \cdots, e^{\nu *}; e_\alpha, \cdots, e_\beta)$

□

生标变换:

$T = T^{\mu \cdots \nu}{}_{\rho \cdots \sigma}\, e_\mu \otimes \cdots \otimes e_\nu \otimes e^{\rho *} \otimes \cdots \otimes e^{\sigma *}$

$= T'^{\alpha \cdots \beta}{}_{\gamma \cdots \delta}\, e'_\alpha \otimes \cdots \otimes e'_\beta \otimes e'^{\gamma *} \otimes \cdots \otimes e'^{\delta *}$

$V^\mu e_\mu = V'^\alpha e'_\alpha = V^\mu A^\alpha{}_\mu e'_\alpha \Rightarrow e_\mu = A^\alpha{}_\mu e'_\alpha$

或 $e'_\alpha = (A^{-1})^\mu{}_\alpha e_\mu$

同理 $e'^{\gamma *} = A^\gamma{}_\rho e^{\rho *}$

$\therefore T^{\mu \cdots \nu}{}_{\rho \cdots \sigma}\, e_\mu \otimes \cdots \otimes e_\nu \otimes e^{\rho *} \otimes \cdots \otimes e^{\sigma *}$

$= T'^{\alpha \cdots \beta}{}_{\gamma \cdots \delta} (A^{-1})^\mu{}_\alpha \cdots (A^{-1})^\nu{}_\beta A^\gamma{}_\rho \cdots A^\delta{}_\sigma\, e_\mu \otimes \cdots \cdots \cdots \otimes e^{\sigma *}$

$\therefore T^{\mu \cdots \nu}{}_{\rho \cdots \sigma} = T'^{\alpha \cdots \beta}{}_{\gamma \cdots \delta} (A^{-1})^\mu{}_\alpha \cdots (A^{-1})^\nu{}_\beta A^\gamma{}_\rho \cdots A^\delta{}_\sigma$

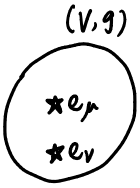
$C^1_1(v \otimes \omega) = \omega(v)$

Pf: $C^1_1(v \otimes \omega) = v \otimes \omega(e^{*\mu}; e_\mu)$

$= v(e^{*\mu})\, \omega(e_\mu) = v^\mu \omega_\mu$

而 $\omega(v) = \omega_\mu v^\nu e^{*\mu}(e_\nu) = \omega_\mu v^\mu$ □

Metric Tensor Field



g is a tensor of type (0,2)

(a) symmetric $\quad g(v,u) = g(u,v)$

(b) nondegenerate

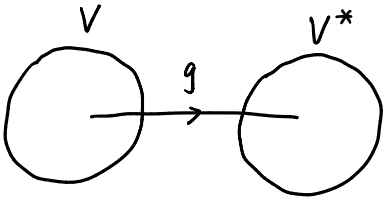
$\forall u \in V,\ g(v,u) = 0 \Rightarrow v = 0 \in V$

定义: $|v| := \sqrt{|g(v,v)|}$

$v \perp u$, iff $g(v,u) = 0$

othonormal basis

$$g_{\mu\nu} = g(e_\mu, e_\nu) = \begin{cases} 0, & \mu \neq \nu \\ \pm 1, & \mu = \nu \end{cases}$$



$g(v,\cdot) \in V^*$

$g : V \xrightarrow{\text{linear}} V^*$

Claim: g is an isomorphism

Pf: $g(\alpha v_1 + \beta v_2, \cdot)$

$= g_{\mu\nu}\, e^{\mu *} \otimes e^{\nu *} (\alpha v_1 + \beta v_2, \cdot)$

$= g_{\mu\nu}\, e^{\mu *}(\alpha v_1^\rho e_\rho + \beta v_2^\sigma e_\sigma)\, e^{\nu *}(\cdot)$

$= \alpha g(v_1, \cdot) + \beta g(v_2, \cdot)$

由于 g 是 undegenerate 且 g 是 linear, 所以

$v \neq u \Rightarrow g(v,\cdot) \neq g(u,\cdot)$, g 是 one-to-one

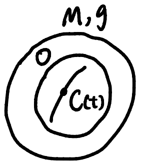
又 g 是 linear map $\therefore$ g 是 onto

$\therefore$ g 是 isomorphism

□

Curve 的长度在重参数化过程中保持不变

Pf: $l = \int \sqrt{g(T,T)}\, dt$



$\frac{\partial}{\partial t} = T^\mu \frac{\partial}{\partial x^\mu} \Rightarrow T^\mu = \frac{dx^\mu}{dt}$, $g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$

$\therefore\ l = \int \sqrt{g_{\mu\nu} T^\mu T^\nu}\, dt$

在重参数化时 $C(t) = C'(t') \Rightarrow t' = t'(t)$

有 $\frac{dt'}{dt} = \frac{\partial C/\partial t}{\partial C'/\partial t'}$

$l = \int \sqrt{g(T',T')}\, dt'$ $T' = \frac{\partial}{\partial t'} = \frac{\partial C'/\partial t'}{\partial C/\partial t} \frac{\partial}{\partial t}$

$\Rightarrow \sqrt{g(T',T')}\, dt' = \sqrt{g(T',T')}\, \frac{\partial C/\partial t}{\partial C'/\partial t'} = \sqrt{g(T,T)}\, dt$

$\therefore \int \sqrt{g(T,T)}\, dt = \int \sqrt{g(T',T')}\, dt'$ □

又 $T^\mu = \frac{dx^\mu}{dt}$ $\therefore l = \int \sqrt{g_{\mu\nu} T^\mu T^\nu}\, dt = \int \sqrt{g_{\mu\nu} dx^\mu dx^\nu}$

令 $ds^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$ $l = \int \sqrt{|ds^2|} = \begin{cases} \int \sqrt{ds^2} & (\text{s.l.}) \\ \int \sqrt{-ds^2} & (\text{t.l.}) \end{cases}$

如果一条 curve 一会儿是 t.l.、一会儿是 s.l. 则 l 无定义

注意: 对比 $ds^2 = g_{\mu\nu} dx^\mu dx^\nu$ 与 $g = g_{\mu\nu} dx^\mu \otimes dx^\nu$

发现 ds^2 只是 g 的另一种写法

The abstract index notation

1. 指标平衡

2. $T^\mu{}_\mu = T(e^{\mu *}; e_\mu) \equiv T^a{}_a$

3. $T \otimes S \equiv T^a{}_{bc} S^d{}_e$

4. $V = V^\mu e_\mu \longrightarrow V^a = V^\mu (e_\mu)^a$
$\omega = \omega_\nu e^{\nu *} \longrightarrow \omega_b = \omega_\nu (e^\nu)_b$ 略去 star
$V^\mu = V(e^{*\mu}) \rightarrow V^\mu = V^a (e^\mu)_a$
$\omega_\nu = \omega(e_\nu) \rightarrow \omega_\nu = \omega_b (e_\nu)^b$

5. $T^a{}_b : V \rightarrow V$, $T^a{}_b V^b \in V$
$T^a{}_b : V^* \rightarrow V^*$, $T^a{}_b \omega_a \in V^*$

$\delta^a{}_b \equiv \delta^\mu{}_\nu (e_\mu)^a (e^\nu)_b$

$\delta^a{}_b V^b = \delta^\mu{}_\nu (e_\mu)^a (e^\nu)_b\ V^\rho (e_\rho)^b$

注: $(e^\nu)_b (e_\rho)^b = e^{*\nu}(e_\rho) = \delta^\nu{}_\rho$

$= \delta^\mu{}_\nu (e_\mu)^a \delta^\nu{}_\rho V^\rho = V^\mu (e_\mu)^a = V^a$

$\delta^a{}_b \omega_a = \delta^\mu{}_\nu (e_\mu)^a (e^\nu)_b\ \omega_\rho (e^\rho)_a$
$= \delta^\mu{}_\nu\ \delta^\rho{}_\mu (e^\nu)_b \omega_\rho$
$= \omega_\nu (e^\nu)_b = \omega_b$

6. $g_{ab} : V \rightarrow V^*$ 或 $V^b \mapsto g_{ab} V^b$

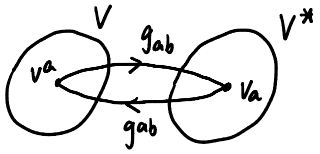
$V_a \equiv g_{ab} V^b$

g 作为 isomorphism，具有 inverse $g^{-1} : V^* \rightarrow V$

g^{-1} is a tensor of type (2,0)

$g^{ab} : V^* \rightarrow V$

$\omega^a = g^{ab} \omega_b$



$V^c = g^{cb}(g_{ba} V^a)$

$\Rightarrow \delta^c{}_a = g^{cb} g_{ba}$

注: 抽象指标: $a\, b\, c \cdots$
具体指标: $\mu\, \nu\, \rho \cdots = 0,1,2,3$
$i, j, k, l = 1,2,3$ ← 不是抽象指标

7. $g_{ab} \left(\frac{\partial}{\partial x^\mu}\right)^b = g_{\mu\nu} (dx^\nu)_a$

Pf: $g_{ab} = g_{\rho\sigma} (dx^\rho)_a (dx^\sigma)_b$

左 $= g_{\rho\sigma} (dx^\rho)_a (dx^\sigma)_b \left(\frac{\partial}{\partial x^\mu}\right)^b$
$= g_{\rho\sigma} (dx^\rho)_a\ \delta^\sigma{}_\mu$
$= g_{\rho\mu} (dx^\rho)_a =$ 左 □

相似地 $g^{ab} (dx^\mu)_b = g^{\mu\nu} \left(\frac{\partial}{\partial x^\nu}\right)^a$
↓
$g^{\rho\sigma} \left(\frac{\partial}{\partial x^\rho}\right)^a \left(\frac{\partial}{\partial x^\sigma}\right)^b$

$$T_{(abc)} = \frac{1}{3!}(T_{abc} + T_{acb} + T_{bac} + T_{cab} + T_{bca} + T_{cba})$$

$$T_{[abc]} = \frac{1}{3!}(T_{abc} - T_{acb} + T_{cab} - T_{bac} + T_{bca} - T_{cba})$$

$$T_{(a_1\cdots a_l)} := \frac{1}{l!}\sum_\pi T_{a_{\pi(1)}\cdots a_{\pi(l)}}$$

$$T_{[a_1\cdots a_l]} := \frac{1}{l!}\sum_\pi \delta_\pi T_{a_{\pi(1)}\cdots a_{\pi(l)}}$$

Claim: ① $T_{a_1\cdots a_l} = T_{(a_1\cdots a_l)}$，则 $T_{a_1\cdots a_n} = T_{a_{\pi(1)}\cdots a_{\pi(l)}}$

② $T_{a_1\cdots a_l} = T_{[a_1\cdots a_l]}$，则 $T_{a_1\cdots a_l} = \delta_\pi T_{a_{\pi(1)}\cdots a_{\pi(l)}}$

Pf: ① $\because T_{a_1\cdots a_l} = T_{(a_1\cdots a_l)}$

$\therefore T_{a_{\pi(1)}\cdots a_{\pi(l)}} = T_{(a_{\pi(1)}\cdots a_{\pi(l)})}$

而 $T_{(a_1\cdots a_l)} = T_{(a_{\pi(1)}\cdots a_{\pi(l)})}$

$\therefore T_{a_1\cdots a_l} = T_{a_{\pi(1)}\cdots a_{\pi(l)}}$

② $\because T_{a_1\cdots a_l} = T_{[a_1\cdots a_l]}$

$\therefore T_{a_{\pi(1)}\cdots a_{\pi(l)}} = T_{[a_{\pi(1)}\cdots a_{\pi(l)}]}$

而 $T_{[a_{\pi(1)}\cdots a_{\pi(l)}]} = \delta_\pi T_{[a_1\cdots a_l]}$

$\therefore T_{a_{\pi(1)}\cdots a_{\pi(l)}} = \delta_\pi T_{a_1\cdots a_l}$ □

Claim: ① $T_{[a_1\cdots a_l]} S^{a_1\cdots a_l} = T_{[a_1\cdots a_l]} S^{[a_1\cdots a_l]}$

$= T_{a_1\cdots a_l} S^{[a_1\cdots a_l]}$ 对 ( ) 一样

② $T_{[a_1\cdots[a_r\cdots a_t]\cdots a_l]} = T_{[a_1\cdots a_l]}$

对 ( ) 一样

③ $T_{[a_1\cdots(a_r\cdots a_t)\cdots a_l]} = 0$

④ 异种括号缩并得 0

⑤ $T_{a_1\cdots a_l} = T_{(a_1\cdots a_l)} \Rightarrow T_{[a_1\cdots a_l]} = 0$

$T_{a_1\cdots a_l} = T_{[a_1\cdots a_l]} \Rightarrow T_{(a_1\cdots a_l)} = 0$

Pf: ① $T_{[a_1\cdots a_l]} S^{a_{\pi(1)}\cdots a_{\pi(l)}} = T_{[a_{\pi(1)}\cdots a_{\pi(l)}]} S^{a_1\cdots a_l}$

$= \delta_\pi T_{[a_1\cdots a_l]} S^{a_1\cdots a_l}$

$\therefore T_{[a_1\cdots a_l]} S^{[a_1\cdots a_l]} = \frac{1}{l!}\sum \delta_\pi^2 T_{[a_1\cdots a_l]} S^{[a_{\pi 1}\cdots a_{\pi l}]}$

②～⑤ 易得 □

Derivative Operator

$(df)_a = \nabla_a f$

Pf: 对于 $\forall\, v^a \in V$

$(df)_a v^a = v(f) = v^a \nabla_a f$ □

对于另一个 derivative operator $\tilde{\nabla}_a$ over M

$\tilde{\nabla}_a f = (df)_a = \nabla_a f$

对于两个 dual vector field ω_b，ω'_b

$\omega_b|_p = \omega'_b|_p$，$(\nabla_a \omega_b)|_p \neq (\nabla_a \omega'_b)|_p$

Claim $[(\tilde{\nabla}_a - \nabla_a)\omega_b]_p = [(\tilde{\nabla}_a - \nabla_a)\omega'_b]_p$

Pf: 令 $\Omega_b = \omega_b - \omega'_b$

$[\nabla_a \Omega_b]_p = [\nabla_a(\Omega_\mu (dx^\mu)_b)]_p$

$= [\underset{\substack{\downarrow \\ 为0}}{\Omega_\mu} \nabla_a (dx^\mu)_b]_p + [(dx^\mu)_b \nabla_a \Omega_\mu]_p$

同理，

$[\tilde{\nabla}_a \Omega_b]_p = [(dx^\mu)_b \tilde{\nabla}_a \Omega_\mu]_p$

Ω_μ 是一个 scalar field，$\nabla_a \Omega_\mu = \tilde{\nabla}_a \Omega_\mu$

□

(如果联系以后面的内容，可知，

$\nabla_a (dx^\mu)_b = \nabla_a(\delta^\mu{}_\nu (dx^\nu)_b) = \nabla_\rho(\delta^\mu{}_\nu (dx^\nu)_b)(dx^\rho)_a$

$= (\delta^\mu{}_{\nu,\rho} - \Gamma^\sigma_{\nu\rho}\delta^\mu{}_\sigma)(dx^\rho)_a (dx^\nu)_b$

$= -\Gamma^\mu_{\nu\rho}(dx^\rho)_a (dx^\nu)_b$

$\therefore \nabla_a$，$\tilde{\nabla}_a$ 是对应不同的仿射联络系数)

p 点上一 dual vector ω_b，它可以扩展为一个 dual vector field (只要在 p 点值为 ω_b 的所有 dual vector field) 无论怎么扩展，与 $\tilde{\nabla}_a - \nabla_a$ 作用，结果唯一

$\therefore \tilde{\nabla}_a - \nabla_a : V_p^* \xrightarrow{\text{linear}} \mathcal{T}_{V_p}(0,2)$ 对应 $C^c_{ab} = \delta^c{}_a \omega_b$

$\therefore \tilde{\nabla}_a - \nabla_a \in \mathcal{T}_{V_p}(1,2)$ $[(\tilde{\nabla}_a - \nabla_a)v_b]_p = C^c_{ab} v_c$

也可为 dual vector

在无挠条件下，$C^c{}_{ab}$ 下两指标对称

Pf: 令 $\nabla_a V_b = \tilde{\nabla}_a V_b - C^c{}_{ab} V_c$

$$\nabla_a \nabla_b f = \tilde{\nabla}_a \underbrace{\nabla_b f}_{=\tilde{\nabla}_b f} - C^c{}_{ab} \underbrace{\nabla_c f}_{=\tilde{\nabla}_c f}$$

$$\nabla_b \nabla_a f = \tilde{\nabla}_b \nabla_a f - C^c{}_{ba} \nabla_c f$$

无挠情况下 $\nabla_a \nabla_b f = \nabla_b \nabla_a f$

则 $C^c{}_{ab} \nabla_c f = C^c{}_{ba} \nabla_c f$ for $\forall f \in \mathscr{F}_M$

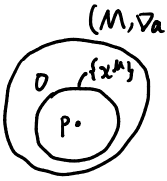
$\therefore C^c{}_{ab} = C^c{}_{ba}$ □

$v^b V_b$ 相当于一个 scalar field $[(\tilde{\nabla}_a - \nabla_a)(v^b V_b)]\big|_p = 0$

$$v^b(\tilde{\nabla}_a - \nabla_a) V_b + V_b(\tilde{\nabla}_a - \nabla_a) v^b = 0$$

$$\Rightarrow v^b C^c{}_{ab} V_c + V_c \delta^c{}_b (\tilde{\nabla}_a - \nabla_a) v^b = 0$$

$$\Rightarrow \nabla_a v^b = \tilde{\nabla}_a v^b + C^b{}_{ac} v^c$$



在 patch O 上，∂_a 是一个 derivative operator

$$\partial_a V_b = (dx^\mu)_a (\partial_\mu V_\nu)(dx^\nu)_b$$

令 $(\nabla_a - \partial_a) V_b = -\Gamma^c{}_{ba} V_c$

$\therefore \nabla_a V_b = \partial_a V_b - \Gamma^c{}_{ba} V_c$

↖ ↑ 这是一个张量(坐标系依赖的)

	$\{x^\mu\}$	$\{x'^\mu\}$
$\{x^\mu\}: \Gamma^c{}_{ab}$	$\Gamma^\sigma{}_{\mu\nu}$	$\Gamma'^\sigma{}_{\mu\nu}$
$\{x'^\mu\}: \bar{\Gamma}^c{}_{ab}$	$\bar{\Gamma}^\sigma{}_{\mu\nu}$	$\bar{\Gamma}'^\sigma{}_{\mu\nu}$

不等

↔ 满足张量分量变换规律

↘ 这两个是传统定义下的克氏符

Commutator

$$[u,v](f) := u(v(f)) - v(u(f))$$

$$[u,v](f) = u^b \nabla_b (v^a \nabla_a f) - v^a \nabla_a (u^b \nabla_b f)$$

↓

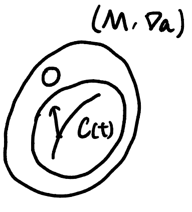
都是 scalar field，与 ∇_a 选取无关

$$= u^b v^a \nabla_b \nabla_a f + u^b (\nabla_b v^a) \nabla_a f$$
$$- v^a u^b \nabla_a \nabla_b f - v^a (\nabla_a u^b) \nabla_b f$$

无挠情况下

$$= (u^b \nabla_b v^a - v^b \nabla_b u^a) \nabla_a f$$

Parallel transport of a vector along a curve



要求：$T^b \nabla_b v^a = 0$

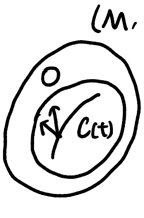
$$左 = T^b(\partial_b v^a + \Gamma^a{}_{cb} v^c)$$

$$= \Big(\underbrace{T^\nu \frac{\partial v^\mu}{\partial x^\nu}}_{\frac{d\,v^\mu(x^\nu(t))}{dt}} + T^\nu \Gamma^\mu{}_{\sigma\nu} v^\sigma\Big)(dx_\mu)^a$$

(注：$u^b v_b = u^\nu v_\nu$)

$$\therefore \frac{dv^\mu}{dt} + T^\nu \Gamma^\mu{}_{\sigma\nu} v^\sigma = 0$$

Derivative operator associated with a metric



$$0 = T^c \nabla_c (g_{ab} v^a u^b)$$

$$= v^a u^b T^c \nabla_c g_{ab} + g_{ab} v^a \underbrace{T^c \nabla_c u^b}_{0}$$
$$+ g_{ab} u^b \underbrace{T^c \nabla_c v^a}_{0}$$

$$= v^a u^b T^c \nabla_c g_{ab}$$

$$\therefore \nabla_c g_{ab} = 0$$

$$\nabla_a (dx^\mu)_b = \partial_a (dx^\mu)_b - \Gamma^c{}_{ba} (dx^\mu)_c$$

$$\downarrow$$

$$(dx^\nu)_a (dx^\rho)_b \, \partial_\nu \delta_\rho{}^\mu = 0$$

$$\nabla_a \left(\frac{\partial}{\partial x^\mu}\right)^b = \partial_a \left(\frac{\partial}{\partial x^\mu}\right)^b + \Gamma^b{}_{ca} \left(\frac{\partial}{\partial x^\mu}\right)^c$$

$$\downarrow$$

$$0$$

$$0 = \nabla_c g_{ab} = \tilde{\nabla}_c g_{ab} - C^d{}_{ac} g_{db} - C^d{}_{bc} g_{ad}$$

$$= \tilde{\nabla}_c g_{ab} - C_{bac} - C_{abc}$$

且 $C^c{}_{ab} = C^c{}_{ba}$，得

$$C_{cab} = \frac{1}{2} \left(\tilde{\nabla}_a g_{cb} + \tilde{\nabla}_b g_{ca} - \tilde{\nabla}_c g_{ab} \right)$$

若 $\tilde{\nabla}_a g_{bc} = 0$，则 $C_{cab} = 0$，∇_a 唯一

但给定 ∇_a，g_{ab} 不唯一

Geodesics

$T^b \nabla_b T^a = 0$ (affinely parameterized geodesic)

$T^\mu (\partial_\mu T^a + \Gamma^a{}_{\sigma\mu} T^\sigma) = 0$ 代入 $T^\mu = \frac{dx^\mu}{dt}$

$$\Rightarrow \frac{dT^a}{dt} + \Gamma^a{}_{\sigma\mu} T^\sigma T^\mu = 0$$

$$\Rightarrow \left(\frac{\partial}{\partial x^\rho}\right)^a \left(\frac{d^2 x^\rho}{dt^2} + \Gamma^\rho{}_{\sigma\mu} \frac{dx^\sigma}{dt} \frac{dx^\mu}{dt} \right) = 0$$

Claim: $T^a \nabla_a T^b = 0 \Rightarrow T'^a \nabla_a T'^b = \alpha(t') T'^b$

Pf: $T'^a = \left(\frac{\partial}{\partial t'}\right)^a = \frac{dt}{dt'} \left(\frac{\partial}{\partial t}\right)^a$

$$T'^a \nabla_a T'^b = T'^a \nabla_a \left(\frac{dt}{dt'} T^b \right)$$

$$= T'^a T^b \nabla_a \left(\frac{dt}{dt'}\right) + \underbrace{T^a}_{} \left(\frac{dt}{dt'}\right)^2 \underbrace{\nabla_a T^b}_{(=0)}$$

$$= T^b \frac{d}{dt'} \left(\frac{dt}{dt'}\right)$$

$$= T'^b \frac{dt'}{dt} \frac{d}{dt'} \left(\frac{dt}{dt'}\right)$$

$$= \frac{d}{dt} \left(\frac{dt}{dt'}\right) T'^b \qquad \square$$

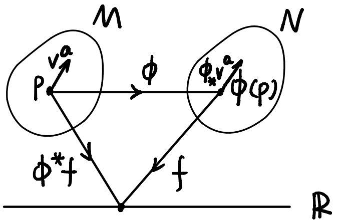
$$\left(\alpha = \frac{d}{dt}\left(\frac{dt}{dt'}\right) \equiv - \left(\frac{d^2 t'}{dt^2}\right) \Big/ \left(\frac{dt'}{dt}\right)^2 \right)$$

$$\downarrow$$

$$\frac{d}{dt}\left(\frac{dt}{dt'} \frac{dt'}{dt}\right) = 0 = \frac{dt'}{dt} \frac{d}{dt}\left(\frac{dt}{dt'}\right) + \frac{dt}{dt'} \frac{d^2 t'}{dt^2}$$

$\alpha = 0$ 则 $t' = at + b$ (affine parameter)

Maps of Manifolds



M, N 是任意 2个 Manifolds

不一定同维

(由于 ϕ^*, ϕ_* 虽为 linear map, 但没有~~onto~~的性质, 所以不对维度作出限制) one-to-one

ϕ is a C^∞ map

$\phi: M \to N$, $(p \mapsto \phi(p))$

1. $\phi^*: \mathcal{F}_N(0,0) \to \mathcal{F}_M(0,0)$ (a Pull back)

$\forall f \in \mathcal{F}_N$, def $\phi^* f \in \mathcal{F}_M$ by

$\phi^* f|_p := f|_{\phi(p)}$, $\forall p \in M$

i.e. $\phi^* f = f \circ \phi$ (易知 ϕ^* is linear)

2. $\phi_*: V_p \to V_{\phi(p)}$ (a Push forward)

$\forall v \in V_p$, def $\phi_* v \in V_{\phi(p)}$ by

$(\phi_* v)(f) := v(\phi^* f)$ (易知 ϕ_* 也为 linear)

i.e. $\overset{N}{\phi_* v} = \overset{M}{v \circ \phi^*}$

$\phi_* v$ is a vector

Pf: ① $(\phi_* v)(\alpha f + \beta g) = v(\phi^*(\alpha f + \beta g))$

$= \alpha v(\phi^* f) + \beta v(\phi^* g)$

$= \alpha (\phi_* v)(f) + \beta (\phi_* v)(g)$

② $(\phi_* v)(fg) = v(\phi^*(fg))$

$= v(\phi^* f \cdot \phi^* g)$

$= \phi^* f|_p \, v(\phi^* g) + \phi^* g|_p \, v(\phi^* f)$

$= f|_{\phi(p)} (\phi_* v)(g) + g|_{\phi(p)} (\phi_* v)(f)$

$\square$

3. $\phi^*: \mathcal{T}_N(0,l) \to \mathcal{T}_M(0,l)$ (Pull back)

$\forall T \in \mathcal{T}_N(0,l)$, def $\phi^* T \in \mathcal{T}_M(0,l)$ by

$$(\phi^* T)_{a_1 \cdots a_l}|_p (v_1)^{a_1} \cdots (v_l)^{a_l} := T_{a_1 \cdots a_l}|_{\phi(p)} (\phi_* v_1)^{a_1} \cdots (\phi_* v_l)^{a_l}$$

$\searrow \in V_p$

4. $\phi_*: \mathcal{T}_{V_p}(k,0) \to \mathcal{T}_{V_{\phi(p)}}(k,0)$ (Push forward)

$\forall T \in \mathcal{T}_{V_p}(k,0)$, def $\phi_* T \in \mathcal{T}_{V_{\phi(p)}}(k,0)$ by

$$(\phi_* T)^{a_1 \cdots a_k} (\omega^1)_{a_1} \cdots (\omega^k)_{a_k} := T^{a_1 \cdots a_k} (\phi^* \omega^1)_{a_1} \cdots (\phi^* \omega^k)_{a_k}$$

$\searrow \in V^*_{\phi(p)}$

? $\phi_*: \mathscr{T}_M(1,0) \to \mathscr{T}_N(1,0)$

$\forall\, v^a \in \mathscr{T}_M(1,0)$, def $\phi_* v^a \in \mathscr{T}_N(1,0)$ by

$$(\phi_* v)^a\big|_{\underbrace{\phi(p)}} \, \underset{\searrow \in V^*_{\phi(p)}}{u_a} := v^a\big|_p \, (\phi^* u)_a$$

由于 ϕ 不是 bijective (仅 C^∞)，导致 N 上同一个点 q，可能 $(\phi_* v)^a$ 有多个值 ($\phi(p_1)=\phi(p_2)=q$)，且 $(\phi_* v)^a$ 不一定铺满 N

但如果 ϕ 是 diffeomorphism，则"?"可行

且对张量的作用不仅限于 $(0,k)$ 与 $(0,l)$ 型的

ϕ 是 diffeomorphism 时，

1. $\phi^*: \mathscr{T}_N(k,l) \to \mathscr{T}_M(k,l)$ (Pull back)

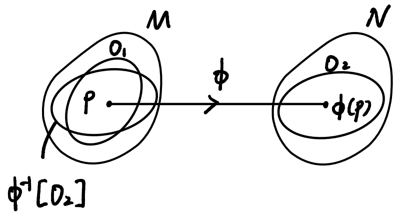
$\forall\, T^{a_1\cdots a_k}{}_{b_1\cdots b_l} \in \mathscr{T}_N(k,l)$, def $(\phi^* T)^{a_1\cdots a_k}{}_{b_1\cdots b_l} \in \mathscr{T}_M(k,l)$

by

$$(\phi^* T)^{a_1\cdots a}{}_{b_1\cdots b_k}\big|_{\underset{\searrow \in M}{p}} \, \underset{\searrow \in V_p^*}{(\omega^1)_{a_1}\cdots(\omega^k)_{a_k}} \, \underset{\searrow \in V_p}{v_1{}^{b_1}\cdots v_l{}^{b_l}}$$

$$:= T^{a_1\cdots a_k}{}_{b_1\cdots b_k}\big|_{\underset{\searrow \in N}{\phi(p)}} \, (\phi_*\omega^1)_{a_1}\cdots\underset{\searrow \in V^*_{\phi(p)}}{(\phi_*\omega^k)_{a_k}} \, (\phi_* v_1)^{b_1}\cdots\underset{\searrow \in V_{\phi(p)}}{(\phi_* v_l)^{b_l}}$$

2. $\phi_*: \mathscr{T}_M(k,l) \to \mathscr{T}_N(k,l)$ (Push forward)



ϕ is a diffeomorphism

$O_1: \{x^\mu\}$, $O_2: \{y^\mu\}$

对于 $\phi^{-1}[O_2]$，坐标 $x'^\mu(p) := y^\mu(\phi(p))$ $\quad p \in O_1 \cap \phi[O_2]$

所以 ϕ 也可以认为是一种坐标变换 $\{x^\mu\} \longmapsto \{x'^\mu\}$

Claim: $T \in \mathscr{T}_{V_p}(k,l)$, $\phi_* T \in \mathscr{T}_{V_{\phi(p)}}(k,l)$

$$T'^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_p = \phi_* T^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_{\phi(p)}$$

Pf: ① $\phi_* v^a\big|_{\phi(p)}(f) = v^a\big|_p(\phi^* f)$, $\big(y^\mu(\phi p) = \underbrace{\phi^* y^\mu}_{x'^\mu}(p)\big)$

$$\Rightarrow \underset{\downarrow\, (\phi_* v)^\mu}{(dy^\mu)_a \, \phi_* v^a\big|_{\phi(p)}} = \underset{\downarrow\, v'^\mu}{(dx'^\mu)_a v^a\big|_p}, \ (\phi^* y^\mu = x^\mu)$$

② $\phi^* w_a\big|_p v^a = w_a\big|_{\phi(p)} \phi_* v^a\big|_{\phi(p)}$

$(\phi^*\omega)_\mu\big|_p v^\mu = \omega_\mu\big|_{\phi(p)} (\phi_* v)^\mu\big|_{\phi(p)}$

$\therefore$ $\omega_\mu\big|_{\phi(p)}$ 在 $\{y^\mu\}$ 上的分量与 $(\phi^*\omega)_\mu\big|_p$ 在 $\{x'^\mu\}$ 上的分量相等

③ $$\phi^* T^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_p \overbrace{(\phi^*\omega^1)_\mu\cdots(\phi^*\omega^k)_\nu\big|_p \, (v_1)^\rho\cdots(v_l)^\sigma}^{\{x'^\mu\}}\big|_p$$

$$= T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta}\big|_{\phi(p)} \underbrace{(\omega^1)_\alpha\cdots(\omega^k)_\beta\big|_{\phi(p)} (\phi_* v_1)^\gamma\cdots(\phi_* v_l)^\delta}_{\{y^\mu\}}\big|_{\phi(p)}$$

$$\Rightarrow \underset{\downarrow\ \text{over } \{x'^\mu\}}{\phi^* T'^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_p} = \underset{\downarrow\ \text{over } \{y^\mu\}}{T^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_{\phi(p)}}$$

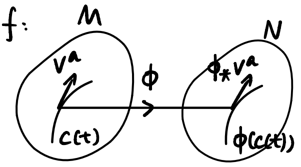
④ $$\phi_* T^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_{\phi(p)} (\omega^1)_\mu\cdots(\omega^k)_\nu\big|_{\phi(p)} (v_1)^\rho\cdots(v_l)^\sigma\big|_{\phi(p)}$$

$$= T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta}\big|_p (\phi^*\omega^1)_\alpha \cdots\cdots (\phi^* v_l)^\delta\big|_p$$

显然 $$\underset{\downarrow\ \text{over } \{y^\mu\}}{\phi_* T^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_{\phi(p)}} = \underset{\downarrow\ \text{over } \{x'^\mu\}}{T'^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\big|_p} \quad \square$$

Another Pf:

①



ϕ 只是 C^∞ map

$$\phi_* T^a(f) = T^a(\phi^* f) = \frac{\partial}{\partial t}\big(f(\phi c(t))\big)$$

正是 $\phi(c(t))$ 的 tangent vector

② $\phi_*\left(\left(\frac{\partial}{\partial x'^\mu}\right)^a\Big|_p\right) = \left(\frac{\partial}{\partial y^\mu}\right)^a\Big|_{\phi(p)}$ 又 $\left(\frac{\partial}{\partial x^\mu}\right)^a (dx^\nu)_a = \delta_\mu{}^\nu$

$$\Rightarrow \phi_*\big((dx'^\mu)_a\big|_p\big) = (dy^\mu)_a\big|_{\phi(p)}$$

③ $$\phi^*\left(T'^{\mu\cdots\nu}{}_{\rho\cdots\sigma}\left(\frac{\partial}{\partial x^{\mu}}\right)^a\cdots\left(\frac{\partial}{\partial x^{\nu}}\right)^b(dx^{\rho})_c\cdots(dx^{\sigma})_d\Big|_p\right)$$

$$= T^{\alpha\cdots\beta}{}_{\gamma\cdots\delta}\left(\frac{\partial}{\partial y^{\alpha}}\right)^a\cdots\left(\frac{\partial}{\partial y^{\beta}}\right)^b(dy^{\gamma})_c\cdots(dy^{\delta})_d\Big|_{\phi(p)}$$

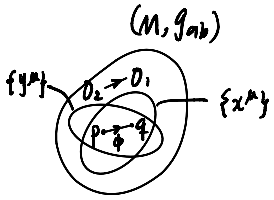
代入②中结论

$$\phi^* T'^{\mu\cdots\nu}{}_{\rho\cdots\sigma} = T^{\mu\cdots\nu}{}_{\rho\cdots\sigma}$$

$\downarrow$ over $\{x'^{\mu}\}$ $\downarrow$ over $\{y^{\mu}\}$ □

Pull back and Push forward

ϕ is a diffeomorphism.



$\phi[O_2] = O_1$ $p, q \in O_1 \cap O_2$

$\phi(p) = q \Rightarrow y^{\mu}(p) = x^{\mu}(q)$

(可以认为 $x^{\mu} = \phi_* y^{\mu}$)

① $(\phi^* f)(p) = f(\phi(p))$ or $h(p) = (\phi_* h)(\phi(p))$

② $v^a(f) = (\phi_* v)^a(\phi_* f)$

令 $f = y^{\mu}$, $\phi_* f = x^{\mu}$

$v^a(dy^{\mu})_a = (\phi_* v)^a(dx^{\mu})_a$ or $v'^{\mu} = (\phi_* v)^{\mu}$

③ $T_{a\cdots b}(v_1)^a\cdots(v_l)^b = (\phi_* T)_{a\cdots b}(\phi_* v_1)^a\cdots(\phi_* v_l)^b$

$$\left(\phi_* \frac{\partial}{\partial y^{\mu}}\right)^a (dx^{\nu})_a = \left(\frac{\partial}{\partial y^{\mu}}\right)^a (dy^{\nu})_a \Rightarrow \left(\phi_* \frac{\partial}{\partial y^{\mu}}\right)^a = \left(\frac{\partial}{\partial x^{\mu}}\right)^a$$

$$T_{a\cdots b}\left(\frac{\partial}{\partial y^{\mu}}\right)^a\cdots\left(\frac{\partial}{\partial y^{\nu}}\right)^b = (\phi_* T)_{a\cdots b}\left(\frac{\partial}{\partial x^{\mu}}\right)^a\cdots\left(\frac{\partial}{\partial x^{\nu}}\right)^b$$

or $T'_{\mu\cdots\nu} = (\phi_* T)_{\mu\cdots\nu}$

$$(dy^{\mu})_a\left(\frac{\partial}{\partial y^{\nu}}\right)^a = (\phi_* dy^{\mu})_a\left(\frac{\partial}{\partial x^{\nu}}\right)^a \Rightarrow (\phi_* dy^{\mu})_a = (dx^{\mu})_a$$

④ $$T^{a\cdots b}{}_{c\cdots d}(dy^{\mu})_a\cdots(dy^{\nu})_b\left(\frac{\partial}{\partial y^{\rho}}\right)^c\cdots\left(\frac{\partial}{\partial y^{\sigma}}\right)^d$$

$$= (\phi_* T)^{a\cdots b}{}_{c\cdots d}(dx^{\mu})_a\cdots(dx^{\nu})_b\left(\frac{\partial}{\partial x^{\rho}}\right)^c\cdots\left(\frac{\partial}{\partial x^{\sigma}}\right)^d$$

Lie derivative of T with respect to $v^a \in \mathscr{F}_M(1,0)$

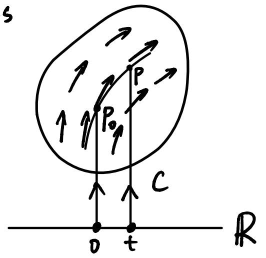
$v^a \in \mathscr{F}_M(1,0)$ 的 intergral curve :

以 v^a 为 tangent vector 的曲线 $C(t)$

one-parameter group of diffeomorphisms

$\phi_t(p_0) = C(t) = p$

$\phi_{t_2} \circ \phi_{t_1}(p_0) = \phi_{t_2}(p_1) = C(t_1+t_2) = p_2$



设 $T^{a_1\cdots a_k}{}_{b_1\cdots b_l} \in \mathscr{F}_M(k,l)$ ，给出 Lie derivative

$$\mathcal{L}_v T^{a_1\cdots a_k}{}_{b_1\cdots b_l} := \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*(T|_{\phi_t(p)})|_p - T|_p\right]$$

adapted coordinate system 适配坐标系：

将 vector field 的 $C(t)$ 视为一个坐标（比如 x^1）$x^1 = t$，其它 $x^2 \cdots x^n$ 与 x^1 横截

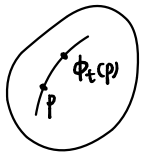
Claim : over an adapted coordinate system

$$(\mathcal{L}_v T)^{\mu\cdots}{}_{\nu\cdots} = \frac{\partial T^{\mu\cdots}{}_{\nu\cdots}}{\partial x^1}$$

Pf: $(\mathcal{L}_v T)^{\mu\cdots}{}_{\nu\cdots} = \lim_{t\to 0}\frac{1}{t}\left[\phi_t^*(T|_{\phi_t(p)})^{\mu\cdots}{}_{\nu\cdots}|_p - T^{\mu\cdots}{}_{\nu\cdots}|_p\right]$

$\{x^\mu\} \xmapsto{\phi_t} \{x'^\mu\}$ $\quad x'^\mu(p) = x^\mu(\phi_t(p))$

$$\underset{\{x'^\mu\}}{\underset{\downarrow}{(\phi_t^* T)^{\mu\cdots}{}_{\nu\cdots}}}\Big|_p = \underset{\{y^\mu\}=\{x^\mu\}}{\underset{\downarrow}{T^{\mu\cdots}{}_{\nu\cdots}}}\Big|_{\phi_t(p)} = \Big(\underset{\{x'^\mu\}}{\underset{\downarrow}{\frac{\partial x'^\mu}{\partial x^\alpha}\cdots\frac{\partial x^\beta}{\partial x'^\nu}}}\cdots T^{\alpha\cdots}{}_{\beta\cdots}\Big)\Big|_{\phi_t(p)}$$



$x'^\mu(p) = x^\mu(\phi_t(p))$

由于 $C(t)$ 是一条坐标轴 x^1

$\therefore x'^1(p) = x^1(\phi_t(p)) = x^1(p) + t$

$x'^\mu(p) = x^\mu(\phi_t(p)) = x^\mu(p)$ $\quad \mu = 2\cdots$

$\therefore \frac{\partial x'^\mu}{\partial x^\nu} = \delta^\mu{}_\nu \qquad \frac{\partial x^\mu}{\partial x'^\nu} = \delta^\mu{}_\nu$

$$\therefore (\phi_t^* T)^{\mu\cdots}{}_{\nu\cdots}|_p = T'^{\mu\cdots}{}_{\nu\cdots}|_{\phi_t(p)}$$

$$\underset{\{x'^\mu\}}{\downarrow} \qquad\qquad \underset{\{x'^\mu\}}{\downarrow}$$

$$\therefore (\mathcal{L}_v T)^{\mu\cdots}{}_{\nu\cdots} = \lim_{t\to 0}\frac{1}{t}\left(T^{\mu\cdots}{}_{\nu\cdots}|_{\phi_t(p)} - T^{\mu\cdots}{}_{\nu\cdots}|_p\right)$$

$$= \left(\frac{\partial}{\partial t}T^{\mu\cdots}{}_{\nu\cdots}\right)\Big|_p \qquad \square$$

Claim: $\mathcal{L}_v u^a = [v,u]^a = v^b\nabla_b u^a \underbrace{- u^b\nabla_b v^a}$ （必须有这一项梳换?）

Pf: 在 adapted coordinate system 中进行证明．（∇_b 为任意的，所以选用 ∂_b）

$[v,u]^\mu = (dx^\mu)_a(v^b\partial_b u^a - u^b\partial_b \underset{(\frac{\partial}{\partial x^1})^a \text{ 与 } \partial_b \text{ 作用后为}0}{\underset{\downarrow}{v^a}})$

$= (dx^\mu)_a v^b\partial_b u^a$

$= (dx^\mu)_a \frac{\partial}{\partial x^1}u^a = \frac{\partial}{\partial x^1}u^\mu$

而由上一个 Claim 知 $\mathcal{L}_v u^\mu = \frac{\partial}{\partial x_1}u^\mu$ $\quad\square$

Claim: $\mathcal{L}_v w_a = v^b\nabla_b w_a + w_b\nabla_a v^b$

$\downarrow$

$v^b\partial_b w_a - \cancel{v^b\Gamma^c{}_{ab}w_c} + w_b\partial_a v^b + \cancel{w_b\Gamma^b{}_{ca}v^c}$

Pf: 在 adapted coordinate system 中，

$v^b\partial_b w_\mu + w_b\partial_\mu v^b \rightarrow (\frac{\partial}{\partial x^1})^b$

$= \frac{\partial}{\partial x^1}w_\mu$

$\mathcal{L}_v w_\mu = \frac{\partial}{\partial x^1}w_\mu$ $\quad \therefore$ 左 = 右 $\quad\square$

Claim: $\mathcal{L}_v T^{a_1\cdots}{}_{b_1\cdots} = v^b\nabla_b T^{a_1\cdots}{}_{b_1\cdots} - (T^{ba_2\cdots}{}_{b_1\cdots}\nabla_b v^{a_1} + \cdots)$

$+ (T^{a_1\cdots}{}_{bb_2\cdots}\nabla_{b_1}v^b + \cdots)$

Pf: $v^b\partial_b T^{\mu_1\cdots}{}_{\nu_1\cdots} + v^b(\Gamma^{\mu_1}{}_{\rho b}T^{\rho\mu_2\cdots}{}_{\nu_1\cdots} + \cdots) - v^b(\Gamma^\rho{}_{\nu_1 b}T^{\mu_1\cdots}{}_{\rho\nu_2\cdots})$

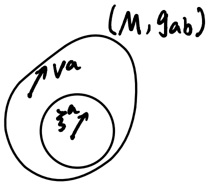
$\updownarrow$

$-(T^{b\mu_2\cdots}{}_{\nu_1\cdots}(\underset{\searrow 0}{\partial_b v^{\mu_1}} + \Gamma^{\mu_1}{}_{\rho b}v^\rho) + \cdots)$

$+(T^{\mu_1\cdots}{}_{b\nu_2\cdots}(\underset{\searrow 0}{\partial_{\nu_1}v^b} + \Gamma^b{}_{\rho\nu_1}v^\rho) + \cdots)$

$= v^b\partial_b T^{\mu_1\cdots}{}_{\nu_1\cdots} = \frac{\partial}{\partial x^1}T^{\mu_1\cdots}{}_{\nu_1\cdots}$ $\quad\square$

Killing vector field



$\phi: M \to M$ 是 diffeomorphism

如果 $\phi^* g_{ab} = g_{ab}$，则称 ϕ 为 isometry $\wedge$ diffeomorphism

$v^a \mapsto \{\phi_t \mid t \in \mathbb{R}\}$. $\phi_t : M \to M$ is a diff.

$\xi^a \mapsto \{\phi_t \mid t \in \mathbb{R}\}$, $\phi_t : M \to M$ is a iso.

↳ one-parameter group of isometries

ξ^a is Killing $\Leftrightarrow \mathcal{L}_\xi g_{ab} = 0 \Leftrightarrow \nabla_{(a}\xi_{b)} = 0 \Leftrightarrow \nabla_a \xi_b = \nabla_{[a}\xi_{b]}$

↳ Killing equations

$0 = \mathcal{L}_\xi g_{ab} = \xi^c \nabla_c g_{ab} + 2 g_{c(b} \nabla_{a)} \xi^c$

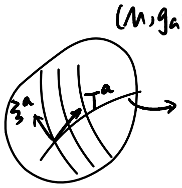
(Choose $\nabla_c g_{ab} = 0$)

$= 2\nabla_{(a}\xi_{b)}$

Claim: $\exists \{x^\mu\}$ s.t. $\frac{\partial g_{\mu\nu}}{\partial x^1} = 0 \Rightarrow (\frac{\partial}{\partial x^1})^a$ is killing

Claim: 沿 geodesic $T^a \xi_a$ 为常数，即 $T^b \nabla_b (\xi_a T^a) = 0$

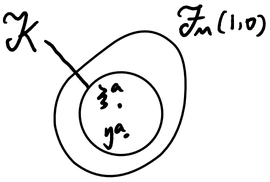
Pf:



→ geodesic $T^b \nabla_b (\xi_a T^a)$

$= T^b T^a \underline{\nabla_b \xi_a} + \xi_a T^b \nabla_b T^a = 0$

antisymmetric ↳ $= 0$



$\alpha \xi^a + \beta \eta^a \in \mathcal{K}$ ($\nabla_{(a}\xi_{b)} = 0$) ↖ 求导线性

$\mathcal{K}$ is a vector space

Claim: $\dim \mathcal{K} \leq \frac{n(n+1)}{2}$ ($\dim \mathcal{K}$ 越高 Riemann space 的对称性越高.)

Example 1. $(\mathbb{R}, \delta_{ab})$

3个独立 Killing vector field 为 $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial \phi}$

$$ds^2 = dx^2 + dy^2 = r^2 d\phi^2 + dr^2$$

$$\frac{\partial}{\partial \phi} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$$

Ex 2. $(\mathbb{R}^3, \delta_{ab})$

6个: $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, -z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}, -x\frac{\partial}{\partial z} + z\frac{\partial}{\partial x}$

Ex 3 $(\mathbb{R}^2, \eta_{ab})$

$ds^2 = -dt^2 + dx^2 = d\psi^2 - \psi^2 d\eta^2$

$x = \psi \cosh\eta$, $t = \psi \sinh\eta$

$\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial \eta} = t\frac{\partial}{\partial x} + x\frac{\partial}{\partial t}$

↳ boost

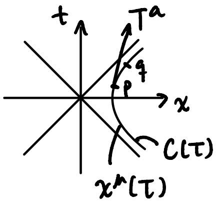
Ex 4. $(\mathbb{R}^4, \eta_{ab})$

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}, \frac{\partial}{\partial t}$ 4

$-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, \cdots$ 3

$x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \cdots$ 3

From boost to Lorenz transformation



$x = \psi\cosh\eta$, $t = \psi\sinh\eta$

$x^2 - t^2 = \psi^2$

$T^a = (\frac{\partial}{\partial \eta})^a \mapsto \{\phi_\tau \mid \tau \in \mathbb{R}\}$

$\frac{\partial}{\partial \eta} = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x} \Rightarrow T^\mu = (x, t)$

又 $T^\mu = \frac{dx^\mu}{d\eta} = (\frac{dt}{d\eta}, \frac{dx}{d\eta})$

$\Rightarrow \frac{dt}{d\eta} = x, \frac{dx}{d\eta} = t \Rightarrow x^\mu(\phi_\tau(p))$

$= x^\mu(p) + \int_0^\tau \frac{dx^\mu}{d\eta} d\eta$

$\phi_\tau(p) = q$, $x^\mu(\phi_\tau(p)) = x'^\mu(p)$

$\therefore x'^\mu(p) = f_{\text{Lorenz}}(x^\mu(p))$

Conformal isometry

ϕ is a conformal isometry

$(\phi^* g)_{ab} = \Omega^2 g_{ab}$, Ω is a nonvanishing function

$(x^\mu(\phi(p)) = y^\mu(p)$ i.e. $x^\mu = \phi_* y^\mu$

$$(\phi^* g)_{ab} \left(\frac{\partial}{\partial y^\mu}\right)^a \left(\frac{\partial}{\partial y^\nu}\right)^b = g_{ab} \left(\frac{\partial}{\partial x^\mu}\right)^a \left(\frac{\partial}{\partial x^\nu}\right)^b$$

$\downarrow$

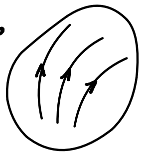
$= \Omega^2 g_{ab} \quad \Rightarrow \quad \Omega^2 g'_{\mu\nu} = g_{\mu\nu})$

conformal Killing vector field ψ^a

$$(\mathcal{L}_\psi g)_{ab} = \psi^c \nabla_c g_{ab} + g_{cb} \nabla^c \psi_a + g_{ac} \nabla^c \psi_b$$

$$= 2\nabla_{(a}\psi_{b)}$$

$$= \alpha g_{ab}$$



$\downarrow$ $\qquad\qquad \uparrow$

$$\lim_{t\to 0} \frac{1}{t} (\phi_t^* g_{\mu\nu} - g_{\mu\nu}) \quad \Uparrow \quad \phi_t^* g_{\mu\nu} = a(t) g_{\mu\nu} \ (a(0) = 1)$$

$$\Rightarrow \quad 2\nabla^c \psi_c = \alpha g^c{}_c = n\alpha \quad \Rightarrow \quad \alpha = \frac{2}{n} \nabla^c \psi_c \ (n = \dim M)$$

$\Rightarrow \quad \nabla_{(a}\psi_{b)} = \frac{1}{n}(\nabla^c \psi_c)$ — conformal Killing equation

Along a geodesic :

$$u^b \nabla_b (u^a \psi_a) = \frac{1}{n} (\nabla^c \psi_c) u^b u_b$$

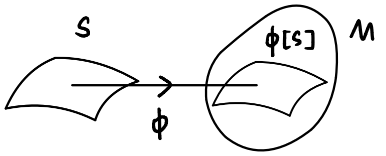
Pf. $u^b \nabla_b (u^a \psi_a)$

$$= \psi_a \underset{\substack{\downarrow \\ 0}}{u^b \nabla_b u^a} + u^a u^b \nabla_b \psi_a$$

$$= u^a u^b \nabla_{(a}\psi_{b)} = u^a u^b \frac{1}{n} (\nabla^c \psi_c) g_{ab} \qquad \square$$

Conformal transformation

Hypersurface



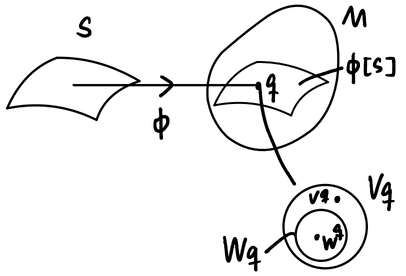
$\dim S \le \dim M$

$\phi: S \to M$ is called an imbedding if ϕ is C^∞, one-to-one and $\phi_*: V_p \to V_{\phi(p)}$ is nondegenerate $\forall p \in S$

$\phi: S \to \phi[S]$ is a diffeomorphism

$\phi[S]$ is called a inbedding submanifold in M

$\dim S = \dim M - 1$ then S is a hypersurface



normal covector (covector 即是 dual vector)

$n_a w^a = 0 \quad \forall w^a \in W_q$

Claim: n_a exists

Pf. W_q 的 basis $\{(e_2)^a \cdots (e_n)^a\}$

则 V_q basis $\{(e_1)^a \cdots (e_n)^a\}$ 且 $(e_1)^a \in V_q - W_q$

对应的 dual bases $\{(e^1)_a \cdots (e^n)_a\}$

则 $n_a = (e^1)_a$ □

若存在 m_a, $m_a (e_\tau)^a = 0$, $(\tau = 2, \cdots n)$, 则 $m_a = \alpha (e^1)_a$

$m_\mu = m_a (e_\mu)^a = m_a (e_1)^a \delta^1{}_\mu$ $\therefore$ m_a 与 n_a 只差常数因子

$f=0$ 可以定义出一个 hypersurface, 前提 $\nabla_a f \big|_{f=c} \neq 0$

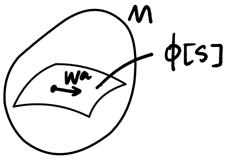
C 为一 constant

$f = C$ 为一个 hypersurface

在这个坐标系下 $\det\{h_{ij}\} = \det\{g_{\mu\nu}\} / n_c n^c$

Claim: $\nabla_a f \big|_{f=0}$ is a normal covector

Pf.



$w^a \nabla_a f = w(f) = 0$ □

现在在 Manifold 上装配 metric field

则 $n^a = n_b g^{ab}$ 是 normal vector

Claim: $n^a \in W_q$, iff $n^a n_a = 0$

Pf. ① ($\Rightarrow$) necessity:

$n^a \in W_q \Rightarrow n^a n_a = 0$

② ($\Leftarrow$) sufficiency:

$n_a = (e^1)_a$, $n^1 = n^a \underbrace{(e^1)_a}_{n_a} = 0$ □

---

induced metric

$V_q, g_{ab} \mapsto W_q, h_{ab}$ ($\phi[S]$ is a inbedding)

$h_{ab} w_1^a w_2^b = g_{ab} w_1^a w_2^b$, $\forall w_1^a, w_2^b \in W_q$

$\phi[S]$ is a hypersurface:

for nonnull hypersurface

$h_{ab} = g_{ab} \mp n_a n_b$

(对于 nonnull hypersurface, 要求 normal vector $n^a n_a = \pm 1$)

Pf. 正交基底 $\{n_a, (e^i)_a\}$, $\{n^a, (e_i)_a\}$ $i, j = 2 \cdots n$

$h_{ab} = h_{ij} (e^i)_a (e^j)_b$

且 $(e^i)_a, (e^j)_b \in W_q$ $\Rightarrow h_{ab} n^a n^b = 0$

$\downarrow$

$h_{\mu\nu} n^\mu n^\nu = 0$

$h_{ab} w^a w^b = g_{ab} w^a w^b$

$h_{ij} = g_{ab} (e_i)^a (e_j)^b$

$h_{ij} = g_{ij} \quad h_{i1} = 0 \quad h_{11} = 0$

$$\begin{cases} g_{11} = g_{ab} n^a n_b = n^a n_a \\ g_{i1} = g_{ab} (e_i)^a n^b = n_a (e_i)^a = 0 \end{cases}$$

$$h_{\mu\nu} = \begin{cases} 0 & , \mu = \nu = 1 \\ 0 = g_{i1} & , \mu \text{ or } \nu = 1 \\ g_{\mu\nu} & , \mu, \nu = 2 \cdots n \end{cases}$$

$\therefore h_{ab} = g_{ab} - n^c n_c \underbrace{n_a n_b}$ ↪ 1,1 分量非零 □

for null hypersurface

$h_{ab} w^a n^b = g_{ab} w^a n^b = 0 \ \forall w^a \in W_q$

↓ degenerate ⇒ 不能作为 metric

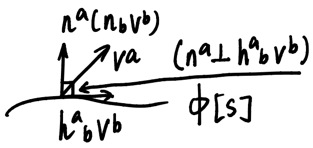
∴ 没有 induced metric

对 non-null hypersurface 作进一步讨论：

$h^a{}_b \equiv g^{ac} h_{cb} = g^{ac}(g_{cb} \mp n_c n_b)$

$= \delta^a{}_b \mp n^a n_b$

$h^a{}_b v^b = v^a \mp n^a n_b v^b \Rightarrow v^a = h^a{}_b v^b \pm n^a(n_b v^b)$



$n_a h^a{}_b v^b = (n_b \mp \overset{\pm 1}{n_a n^a} n_b) v^b$

$= 0$

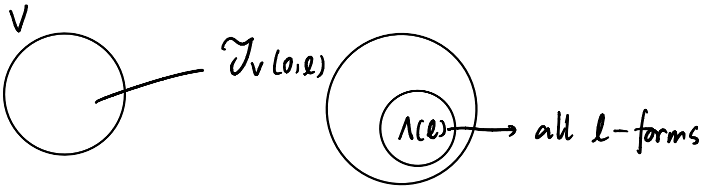
$h^a{}_b$ is a Projection map

投影

Differential forms

if $\omega_{a_1 \cdots a_l} = \omega_{[a_1 \cdots a_l]}$. then it's called l-form

$\Leftrightarrow \omega_{\mu_1 \cdots \mu_l} = \omega_{[\mu_1 \cdots \mu_l]}$



$\Lambda(l)$ is a vector subspace 易证

Wedge Product :

[ ]定义要除以重排数量，用以抵消这一项

$\underline{\omega} \wedge \underline{\mu} := \frac{(l+m)!}{l!\,m!} \omega_{[a_1 \cdots a_l} \mu_{b_1 \cdots b_m]} = \sum_\pi \delta_\pi \omega_{c\pi(1) \cdots} \mu_{\cdots c\pi(l+m)}$

↓ 表示略去 abstract index

↓ 所有有效重排

（比如 $\underline{a_2 a_1} a_3 \cdots a_l ; b_1 \cdots b_m$ 就是无效重排.）

$\mu \wedge \omega = \pm \omega \wedge \mu$

↓ $(-1)^{l \cdot m}$

Claim : $\dim \Lambda(l) = \frac{n!}{l!(n-l)!}$, $l \le n$ （独立分量个数）

Pf. $\omega_{a_1 \cdots a_l} = \omega_{\mu_1 \cdots \mu_l} (e^{\mu_1})_{a_1} \cdots (e^{\mu_l})_{a_l}$

$= \omega_{[\mu_1 \cdots \mu_l]} (e^{[\mu_1})_{a_1} \cdots (e^{\mu_l]})_{a_l}$

$= \omega_{\mu_1 \cdots \mu_l} (e^{\mu_1})_{[a_1} \cdots (e^{\mu_l})_{a_l]}$

$= \frac{1}{l!} \omega_{\mu_1 \cdots \mu_l} (e^{\mu_1})_{a_1} \wedge \cdots \wedge (e^{\mu_l})_{a_l}$

非零项有 $n \times \cdots \times (n-l+1)$ 个，（即 $\frac{n!}{(n-l)!}$ 个），其中，每一项都另有 $l!-1$ 项与之相等

故，独立个数有 $\frac{n!}{l!(n-l)!}$

$\Rightarrow \dim \Lambda(l) = \frac{n!}{l!(n-l)!}$ □

在 manifold 上

$\omega_{a_1 \cdots a_l} = \omega_{\mu_1 \cdots \mu_l} (dx^{\mu_1})_{[a_1} \cdots (dx^{\mu_l})_{a_l]}$

$= \frac{1}{l!} \omega_{\mu_1} \cdots \omega_{\mu_l} (dx^{\mu_1})_{a_1} \wedge \cdots \wedge (dx^{\mu_l})_{a_l}$

$\omega_{\mu_1 \cdots \mu_l} = \omega_{a_1 \cdots a_l} (\frac{\partial}{\partial x^{\mu_1}})^{a_1} \cdots (\frac{\partial}{\partial x^{\mu_l}})^{a_l}$

---

exterior differentiation

$\Lambda_M(l)$, a set of all the l-form fields

$d : \Lambda_M(l) \rightarrow \Lambda_M(l+1)$

def : $(d\omega)_{b a_1 \cdots a_l} := (l+1) \nabla_{[b} \omega_{a_1 \cdots a_l]}$

$\Big(\tilde{\nabla}_{[b} \omega_{a_1 \cdots a_l]} + \overset{\nearrow 0}{C^c{}_{\underset{\sim}{[b a_1}} \omega_{|c|a_2 \cdots a_l]}} + \overset{\nearrow 0}{C^c{}_{\underset{\sim}{[b a_2}} \omega_{a_1 c a_3 \cdots a_l]}}$

$+ \cdots$

$= \tilde{\nabla}_{[b} \omega_{a_1 \cdots a_l]}$, 与 derivative operator 无关 $\Big)$

$l = 0$ 时 $(df)_b = \nabla_b f$

△ $(dx^{\mu_1})_{a_1} \wedge \cdots \wedge (dx^{\mu_l})_{a_l} = (dx^{\mu_1})_{[a_1} \wedge \cdots \wedge (dx^{\mu_{(l-1)}})_{a_{(l-1)}} (dx^{\mu_l})_{a_l]} \overset{\nearrow l}{\frac{l!}{(l-1)!\,1!}}$

$= l!\, (dx^{\mu_1})_{[a_1} \cdots (dx^{\mu_l})_{a_l]} = e_{\mu_1 \cdots \mu_l} (dx^{\mu_1})_{a_1} \cdots (dx^{\mu_l})_{a_l}$

Claim:

$$(d\omega)_{ba_1\cdots a_l} = \sum_C (d\underbrace{\omega_{\mu_1\cdots\mu_l}}_{\text{scalar field}})_b \wedge (dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$$

Pf. left $= (l+1)\nabla_{[b}\omega_{a_1\cdots a_l]}$

$$= (l+1)\nabla_{[b}\omega_{\mu_1\cdots\mu_l}(dx^{\mu_1})_{a_1}\cdots(dx^{\mu_l})_{a_l]}$$

$$= (l+1)\frac{1}{(l+1)!}\nabla_b\omega_{\mu_1\cdots\mu_l}\wedge(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$$

$$= \frac{1}{l!}\nabla_b\omega_{\mu_1\cdots\mu_l}\wedge(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$$

每个组合中有l!重排重复

$$= \sum_C \nabla_b\omega_{\mu_1\cdots\mu_l}\wedge(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_l})_{a_l}$$

□

$$[d(d\omega)]_{cba_1\cdots a_l} = (l+2)(l+1)\nabla_{[c}\nabla_b\omega_{a_1\cdots a_l]}$$

$$= (l+2)(l+1)\partial_{[c}\partial_b\omega_{a_1\cdots a_l]} = 0$$

⇒ $d\circ d = 0$ (无论是否无挠都成立)

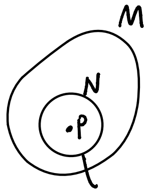
$\underline{\omega}$ is closed if $d\underline{\omega} = 0$,

$\underline{\omega}$ is exact if $\exists\ \underline{\mu}$ s.t. $\underline{\omega} = d\underline{\mu}$

exact ⇒ closed

⇐ ? ↘ locally right, 如果要 globally right, 需要一些条件

(for $\mathbb{R}^n$, globally right)



$d\omega = 0$

$\Rightarrow \underline{\omega} = d\underline{\mu}$

μ在 neighborhood上存在

(p任意)

在q的 neighborhood也存在

但与p的不一样

Integration

仅在 orientable manifold 上讨论 integration

Def. An n-dim manifold M is said to be orientable if $\exists$ a $\underset{\sim}{C^0}$, $\underbrace{\text{nowhere vanishing}}_{\text{处处非零}}$ n-form field on M

(相当于一个 Levi-Civita tensor)

例 $\mathbb{R}^n$ $\varepsilon_{a_1\cdots a_n} = (dx^1)_{a_1}\wedge\cdots\wedge(dx^n)_{a_n}$

unorientable manifold 比如 Möbius strip

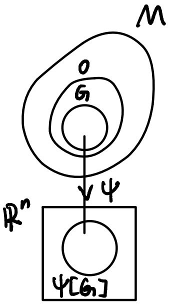
p点 n-form $\dim\Lambda(n) = 1$, $\varepsilon' = h\varepsilon$ h是scalar field

如果 $h>0$ (处处为正), 则给出一样的 orientation

由定义, h是 C^0 且 nowhere vanishing

∴ h要么处处为正, 要么处处为负

(M是连通manifold)



$G\in O$

$$\underline{\omega} = \omega_{1\cdots n}(x^1,\cdots x^n)\,dx^1\wedge\cdots\wedge dx^n$$

$$\int_G \underline{\omega} := \int_{\psi[G]}\omega_{1\cdots n}(x^1,\cdots x^n)\,dx^1\cdots dx^n$$

Claim: $\displaystyle\int_{\psi[G]}\omega_{1\cdots n}(x^1\cdots x^n)\,dx^1\cdots dx^n = \int_{\psi'[G]}\omega'_{1\cdots n}(x'^1\cdots x'^n)\,dx'^1\cdots dx'^n$

Pf. $\displaystyle\omega'_{1\cdots n} = \frac{\partial x^{\mu_1}}{\partial x'^1}\cdots\frac{\partial x^{\mu_n}}{\partial x'^n}\,\omega_{\mu_1\cdots\mu_n}$

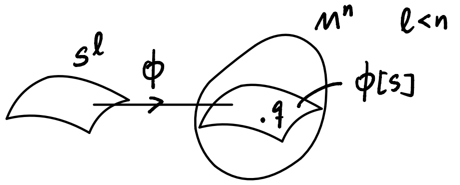
$$= \underbrace{\det\left\{\frac{\partial x^\mu}{\partial x'^\nu}\right\}}_{\text{Jacobi determinant}}\omega_{1\cdots n} \equiv J\,\omega_{1\cdots n}$$

$$\int_{\psi'[G]}\omega'_{1\cdots n}\,dx'^1\cdots dx'^n = \int_{\psi'[G]}\omega_{1\cdots n}\,J\,dx'^1\cdots dx'^n$$

$$= \int_{\psi[G]}\omega_{1\cdots n}\,dx^1\cdots dx^n = 左$$

□

如果右手系换左手系, $J\to -J$, 变号



$W_{a_1 \cdots a_l}$ 在 $\phi[S]$ 内，则 $W_{a_1 \cdots a_l} V^{a_i} = 0$, $i \in \{1, \cdots l\}$,

$\forall\ V^{a_i l} \in V_q - W_q$

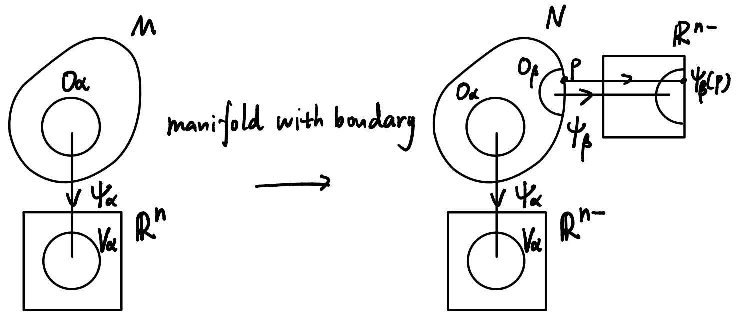
$\underline{w} \mapsto \underline{\tilde{w}}$, $\underline{w} \in V_q - W_q$, $\underline{\tilde{w}} \in W_q$

$\tilde{W}_{a_1 \cdots a_l} (W_1)^{a_1} \cdots (W_l)^{a_l} := W_{a_1 \cdots a_l} (W_1)^{a_1} \cdots (W_l)^{a_l}$

$\underline{\tilde{w}}$ is the restriction of $\underline{w}$

$$\int_{\phi[S]} \underline{w} \equiv \int_{\phi[S]} \underline{\tilde{w}}$$

Manifold with boundary



$R^{n-} := \{(x^1, \cdots x^n) \mid x^1 \le 0\}$

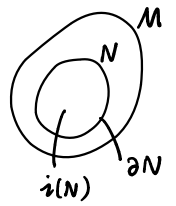
$\partial N \equiv \dot{N}$ the boundary of N

$i(N) \equiv N - \partial N$ the interior of N

$N = i(N) \cup \partial N$

∂N is an $n-1$ dimensional manifold

Stokes theory



M is an oriented manifold.

N is a compact subset of M which is also a n-dim manifold with boundary.

$(n-1)$- form $\underline{w}$ on M

$$\int_{i(N)} d\underline{w} = \int_{\partial N} \underline{w} \equiv \int_{\partial N} \underline{\tilde{w}}$$

$$\downarrow$$

$$\int_{i(N)} \frac{1}{(n-1)!} \nabla_b\, W_{\mu_1 \cdots \mu_{(n-1)}} \wedge (dx^{\mu_1})_{a_1} \wedge \cdots \wedge (dx^{\mu_{(n-1)}})_{a_{(n-1)}}$$

$\downarrow$ 每种组合只算一次

$$= \int_{i(N)} \frac{1}{n!} \nabla_{[\nu_1} W_{\nu_2 \cdots \nu_n]} (dx^{\nu_1})_b \wedge (dx^{\nu_2})_{a_1} \wedge \cdots \wedge (dx^{\nu_n})_{a_{(n-1)}}$$

$$= \int_{i(N)} \frac{1}{n!} \nabla_{[\nu_1} W_{\nu_2 \cdots \nu_n]}\, n!\, (dx^{[\nu_1})_b (dx^{\nu_2})_{a_1} \cdots (dx^{\nu_n]})_{a_{(n-1)}}$$

$$= \int_{i(N)} \nabla_{[\nu_1} W_{\nu_2 \cdots \nu_n]} (dx^{\nu_1})_b (dx^{\nu_2})_{a_1} \cdots (dx^{\nu_n})_{a_{(n-1)}}$$

Volumn element

the signature of a matrix → 相似对角化后，对角元中为负的元素个数

$(-1)^s \rightarrow$ signature

用 $e_{\mu_1 \cdots \mu_n} = \pm 1 = {}^{V}e^{\mu_1 \cdots \mu_n}$ $(g>0)$

$e_{\mu_1 \cdots \mu_n} = \pm 1 = -e^{\mu_1 \cdots \mu_n}$ $(g<0)$

$\varepsilon_{\mu_1 \cdots \mu_n} = \alpha\, e_{\mu_1 \cdots \mu_n}$

$$\varepsilon^{\nu_1 \cdots \nu_n} = g^{\nu_1 \mu_1} \cdots g^{\nu_n \mu_n} \alpha\, e_{\mu_1 \cdots \mu_n}$$

$$= \alpha\, e^{\nu_1 \cdots \nu_n} \det\{g^{\nu\mu}\}$$

且 $\det\{g^{\nu\mu}\} = \det\{g_{\mu\nu}\}^{-1} \equiv g^{-1}$

$\varepsilon^{\nu_1 \cdots \nu_n} = \alpha\, g^{-1} e^{\nu_1 \cdots \nu_n}$

坐标变换

$$\varepsilon'^{\nu_1 \cdots \nu_n} = \frac{\partial x'^{\nu_1}}{\partial x^{\mu_1}} \cdots \frac{\partial x'^{\nu_n}}{\partial x^{\mu_n}} \alpha\, g^{-1} e^{\mu_1 \cdots \mu_n}$$

$$= \alpha\, g^{-1} e^{\nu_1 \cdots \nu_n} \det\left\{\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right\}$$

$$\varepsilon'_{\nu_1 \cdots \nu_n} = \frac{\partial x^{\mu_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\mu_n}}{\partial x'^{\nu_n}} \alpha\, e_{\mu_1 \cdots \mu_n}$$

$$= \det\left\{\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right\}^{-1} \alpha\, e_{\mu_1 \cdots \mu_n}$$

而 $g'_{\nu_1 \nu_2} = \frac{\partial x^{\mu_1}}{\partial x'^{\nu_2}} \frac{\partial x^{\mu_2}}{\partial x'^{\nu_2}} g_{\mu_1 \mu_2} \Rightarrow g' = \det\left\{\frac{\partial x'^{\nu}}{\partial x^{\mu}}\right\}^{-2} g$

代入 $\varepsilon'^{\nu_1\cdots\nu_n} = \alpha g^{-1} e^{\nu_1\cdots\nu_n}\sqrt{\frac{g}{g'}}$

$$= \frac{1}{\sqrt{|g'|}}\ \frac{\alpha}{\sqrt{|g|}}\ e^{\nu_1\cdots\nu_n}$$

$$\varepsilon'_{\nu_1\cdots\nu_n} = \sqrt{\frac{g'}{g}}\ \alpha\, e_{\mu_1\cdots\mu_n}$$

为了具有合理の变换 $\alpha = \sqrt{|g|}$

$\Rightarrow \varepsilon^{\nu_1\cdots\nu_n} = \alpha g^{-1} e^{\nu_1\cdots\nu_n} = \frac{1}{\sqrt{|g|}} e^{\nu_1\cdots\nu_n}$

$\varepsilon_{\nu_1\cdots\nu_n} = \alpha e_{\nu_1\cdots\nu_n} = \sqrt{|g|}\ e_{\nu_1\cdots\nu_n}$

$$\int \varepsilon_{a_1\cdots a_n} = \int \sqrt{|g|}\ e_{\nu_1\cdots\nu_n} (dx^{\nu_1})_{a_1}\cdots(dx^{\nu_n})_{a_n}$$

$$= \int \sqrt{|g|}\ n!\ (dx^{\nu_1})_{[a_1}\cdots(dx^{\nu_n})_{a_n]}$$

$$= \int \sqrt{|g|}\ (dx^{\nu_1})_{a_1}\wedge\cdots\wedge(dx^{\nu_n})_{a_n}$$

Claim: $\nabla_b \varepsilon_{a_1\cdots a_n} = 0$

Pf. $\nabla_b \varepsilon_{a_1\cdots a_n} = \nabla_b\left(\sqrt{|g|}\ e_{\mu_1\cdots\mu_n}(dx^{\mu_1})_{a_1}\cdots(dx^{\mu_n})_{a_n}\right)$

且 $\nabla_b (dx^{\mu})_a = 0$, $\nabla_c g_{ab} = 0 \Rightarrow \nabla_b\sqrt{|g|} = 0$

∴ 为 0 □

Claim: $\varepsilon^{a_1\cdots a_j a_{j+1}\cdots a_n}\varepsilon_{a_1\cdots a_j b_{j+1}\cdots b_n}$

$= (-1)^s (n-j)!\, j!\ \delta^{[a_{j+1}}{}_{b_{j+1}}\cdots\delta^{a_n]}{}_{b_n}$

Pf. left $= e^{a_1\cdots a_j a_{j+1}\cdots a_n} e_{a_1\cdots a_j b_{j+1}\cdots b_n}$

若 $b_{j+1}\cdots b_n$ 是 $a_{j+1}\cdots a_n$ 的偶排列 left $= j!\,(-1)^s$

若 … 是 … 的奇排列 left $= -j!\,(-1)^s$

其它情况 left $= 0$

right $= (-1)^s j!\,(n-j)!\ \delta^{[a_{j+1}}{}_{b_{j+1}}\cdots\delta^{a_n]}{}_{b_n}$

若 $a_{j+1}\cdots a_n$ 是 $b_{j+1}\cdots b_n$ の偶排列

right $= (-1)^s j!\,(n-j)!\ \delta^{[b_{j+1}}{}_{b_{j+1}}\cdots\delta^{b_n]}{}_{b_n} = (-1)^s j!$

若 … 是奇排列 right $= -(-1)^s j!$

others right $= 0$ □

Gauss's law

$$\int_M f := \int_M f\,\underline{\varepsilon}$$

Claim: $\int_{i(N)} (\nabla_b V^b)\,\underline{\varepsilon} = \int_{\partial N} V^b \varepsilon_{b a_1\cdots a_{(n-1)}}$

Pf. 左 $= \int_{\partial N} V^b \varepsilon_{b a_1\cdots a_{(n-1)}} = \int_{i(N)} d\, V^b \varepsilon_{b a_1\cdots a_{(n-1)}}$

$$= \int n\left(\nabla_{[c} V^b \varepsilon_{|b| a_1\cdots a_{(n-1)}]}\right)$$

$$= \int n\left(\nabla_{[c} V^b\right)\varepsilon_{|b| a_1\cdots a_{n-1}]}$$

$$= \int n \nabla_{[\nu} V^\sigma \varepsilon_{|\sigma|\mu_1\cdots\mu_{n-1}]}(dx^\nu)_c\wedge(dx^{\mu_1})_{a_1}\wedge\cdots\wedge(dx^{\mu_{n-1}})_{a_{n-1}}$$

$\downarrow$

非零项为 $\nu\,\mu_1\cdots\mu_{n-1}$ 为 $1\cdots n$ の重排

且 $\sigma = \nu$

$$= \int \frac{n}{n!}\sum_{\nu\mu_1\cdots\mu_n} \overset{\operatorname{sgn}\begin{pmatrix}1\cdots & n\\ \nu\cdots & \mu_{n-1}\end{pmatrix}}{\overset{/\!/}{\delta_\pi}}\left(\nabla_\nu V^\nu \sqrt{|g|}\ e_{\nu\mu_1\cdots\mu_{n-1}}\right)(dx^\nu)_c\wedge\cdots\wedge(dx^{\mu})_a\cdots$$

$$= \int \frac{n}{n!}\sum_{\nu=1}^{n}\Big(\overset{\operatorname{sgn}\begin{pmatrix}1\cdots\cdots & n\\ \nu\, 1\cdots\cdots & n\end{pmatrix}}{\overset{/\!/}{\delta_\pi}}\nabla_\nu V^\nu\sqrt{|g|}\ \underset{\underset{\operatorname{sgn}\begin{pmatrix}1\cdots n\\ \nu\cdots n\end{pmatrix}}{\|}}{e_{\nu 1\cdots \nu-1\,\nu+1\cdots n}}\,(n-1)!\Big)(dx)\wedge\cdots$$

$$= \int \nabla_\nu V^\nu \sqrt{|g|}\ (dx^\nu)_c\wedge\cdots\wedge(dx^{\mu_{n-1}})_{a_{n-1}}$$

$$= \int \nabla_b V^b\ \varepsilon_{c a_1\cdots a_{n-1}}$$ □

Another Pf:

左 $= \int n \nabla_{[c} V^b \varepsilon_{b a_1\cdots a_{n-1}]}$

$n \nabla_{[c} V^b \varepsilon_{|b| a_1\cdots a_{n-1}]} = h\ \varepsilon_{c a_1\cdots a_{n-1}}$

h → scalar field

→ 1-dim 性质

$\varepsilon^{c a_1\cdots a_{n-1}}\ h\ \varepsilon_{c a_1\cdots a_{n-1}} = h(-1)^s n!$

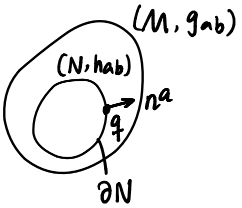
$\varepsilon^{[c a_1\cdots a_{n-1}]}\ n\nabla_{[c} V^b \varepsilon_{|b| a_1\cdots a_{n-1}]}$ ⟶ 可去掉 [ ]

$= n \nabla_c V^b\ (-1)^s\ \delta^c{}_b\ (n-1)!$

$= n!\nabla_b V^b (-1)^s$

∴ $h = \nabla_b V^b$ ∴ 左 $= \int \nabla_b V^b\ \varepsilon_{c a_1\cdots a_{n-1}}$

□



在∂N上的适配体元 $\hat{\varepsilon}_{a_1\cdots a_{n-1}}$

$$\hat{\varepsilon}^{a_1\cdots a_{n-1}} = h^{a_1 b_1}\cdots h^{a_{n-1} b_{n-1}}\hat{\varepsilon}_{b_1\cdots b_{n-1}}$$

$$\hat{\varepsilon}^{a_1\cdots a_{n-1}}\hat{\varepsilon}_{a_1\cdots a_{n-1}} = (-1)^{\hat{s}}(n-1)!$$

Claim: $\int \nabla_b v^b \underline{\varepsilon} = \pm\int_{\partial N} v^b n_b \underline{\hat{\varepsilon}}$, (∂N是 nonnull hypersurface)

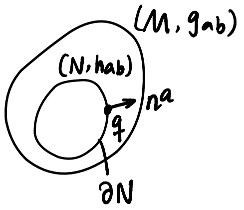
Pf. 等价于证明 $\pm v^b n_b \underline{\hat{\varepsilon}} = \widetilde{(v^b \varepsilon_{b a_1\cdots a_{n-1}})}$

$$\widetilde{(v^b \varepsilon_{b a_1\cdots a_{n-1}})} \equiv \tilde{\omega}_{a_1\cdots a_{n-1}}$$

在∂N上 $(n-1)$-form 是 1-dim

$\therefore\ \tilde{\omega}_{a_1\cdots a_{n-1}} = k\, v^b n_b\, \hat{\varepsilon}_{a_1\cdots a_{n-1}}$

下证 $k=\pm1$



$\dim V_q = n$

选 n 个 basis $\{(e_0)^a \equiv n^a, (e_\mu)^a\}$ $\underset{1\cdots n-1}{\downarrow}$

且是 orthonormal

$\therefore\ \{(e_\mu)^a\}$ 是 W_q 的 basis (W_q 是 ∂N 的 tangent space)

$$k v^b n_b \hat{\varepsilon}_{a_1\cdots a_{n-1}}(e_{\mu_1})^{a_1}\cdots(e_{\mu_{n-1}})^{a_{n-1}} = k v^b n_b \hat{\varepsilon}_{\mu_1\cdots\mu_{n-1}}$$

$(e^0)_a (e_\mu)^a = \delta^0{}_\mu$ 由于 orthonormal

$(e^0)_a = \pm n_a$ $\left(n_a n^a = \pm1 \begin{cases} \text{s.l.} \\ \text{t.l.}\end{cases}\right)$

$$= \pm k v^b (e^0)_b \hat{\varepsilon}_{\mu_1\cdots\mu_{n-1}} = \pm k v^0 \hat{\varepsilon}_{\mu_1\cdots\mu_{n-1}}$$

在 orthonormal basis 下 $h_{\mu\nu} = \delta_{\mu\nu}$, $\hat{\varepsilon}_{\mu_1\cdots\mu_{n-1}} = e_{\mu_1\cdots\mu_{n-1}}$

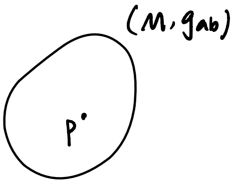
$$\tilde{\omega}_{a_1\cdots a_{n-1}}(e_{\mu_1})^{a_1}\cdots(e_{\mu_{n-1}})^{a_{n-1}} = \omega_{a_1\cdots a_{n-1}}(e_{\mu_1})^{a_1}\cdots(e_{\mu_{n-1}})^{a_{n-1}}$$

$$= v^\nu \varepsilon_{\nu\mu_1\cdots\mu_{n-1}} = v^0 \varepsilon_{0\mu_1\cdots\mu_{n-1}} \xrightarrow{\text{orthonormal}} = v^0 e_{\mu_1\cdots\mu_{n-1}}$$

$\mu_1\cdots\mu_{n-1}$ 是 $1\cdots n-1$ 之重排，所以 $\nu=0$ 才不为0

$$\pm k v^0 e_{\mu_1\cdots\mu_{n-1}} = v^0 e_{\mu_1\cdots\mu_{n-1}} \Rightarrow k=\pm1 \quad \square$$

Dual differential forms



$$\dim\Lambda_p(l) = \frac{n!}{l!(n-l)!} = \dim\Lambda(n-l)$$

$$*: \Lambda_M(l) \to \Lambda_M(n-l)$$

$\forall\, \omega_{a_1\cdots a_l} \in \Lambda_M(l)$, def $*\omega_{a_1\cdots a_{n-l}}$ by

$$(*\omega)_{a_1\cdots a_{n-l}} := \frac{1}{l!}\omega^{b_1\cdots b_l}\varepsilon_{b_1\cdots b_l a_1\cdots a_{n-l}}$$

e.g. $(*f)_{a_1\cdots a_l} = f\varepsilon_{a_1\cdots a_l}$

$$*(*(\omega))_{a_1\cdots a_l}$$

$$= \frac{1}{(n-l)!}\left(\frac{1}{l!}\omega_{b_1\cdots b_l}\varepsilon^{b_1\cdots b_l c_1\cdots c_{n-l}}\right)\varepsilon_{c_1\cdots c_{n-l} a_1\cdots a_l}$$

$$= \frac{1}{(n-l)!\,l!}\omega_{b_1\cdots b_l}(-1)^{l(n-l)}(-1)^s l!\,(n-l)!\,\delta^{[b_1}{}_{a_1}\cdots\delta^{b_l]}{}_{a_l}$$

$$= (-1)^{l(n-l)}(-1)^s\,\omega_{a_1\cdots a_l}$$

back to Euclidian

(1) $(\vec{A}\times\vec{B})_c = {}^*(A\wedge B)_c = \frac{1}{2}\varepsilon_{abc}A^a\wedge B^b$

$$= \frac{1}{2}\varepsilon_{abc}\,2A^{[a}B^{b]} = \varepsilon_{abc}A^aB^b$$

(2) $\operatorname{grad} f = df$

$$\overset{\nearrow 3\delta^d{}_c \nwarrow}{e_{cab}\nabla^c e^{abd}A_d}$$

$$\operatorname{div}\vec{A} = *(d^*\underline{A})\ \left(= \frac{1}{3!}e_{cab}2\nabla^{[c}e^{ab]d}A_d = \nabla^c A_c\right)$$

$$\operatorname{curl}\vec{A} = *(d\underline{A})\ \left(= \frac{1}{2!}e_{abc}2\nabla^{[b}A^{c]} = e_{abc}\nabla^b A^c\right)$$

Claim: ① $\operatorname{curl}\vec{E}=0 \Rightarrow \exists\phi$ s.t. $\vec{E} = \operatorname{grad}\phi$

② $\operatorname{div}\vec{B}=0 \Rightarrow \exists A$ s.t. $\vec{B} = \operatorname{curl} A$

Pf. ① $\operatorname{curl}\vec{E}=0 \Leftrightarrow *(d\underline{E})=0 \Rightarrow d\underline{E}=0$

$\Rightarrow \exists\phi$ s.t. $\underline{E} = d\phi \Rightarrow \vec{E} = \vec{\nabla}\phi$

(用定理)

↖ 难证

② $\operatorname{div}\vec{B}=0 \Leftrightarrow d^*\underline{B}=0 \Rightarrow \exists\,\underline{A}$ s.t. $^*B = d\underline{A}$

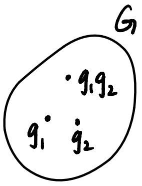
$\Rightarrow B = {}^*(dA) \Rightarrow \vec{B} = \vec{\nabla}\times\vec{A}$

如 ∂N is a null hypersurface, 则 Claim 仍成立，但 $\underline{\hat{\varepsilon}}$ 之定义要作一修改

$\frac{1}{n}\varepsilon_{a_1\cdots a_n} = n_{[a_1}\hat{\varepsilon}_{a_2\cdots a_n]}$ 证略

Lie Groups & Lie Algebras

ABC of Group Theory



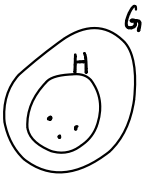
$G \times G \longrightarrow G$

(a) $(g_1 g_2) g_3 = g_1 (g_2 g_3)$

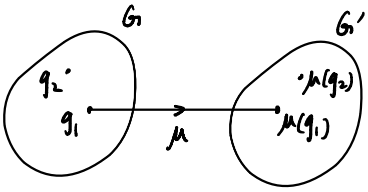
(b) $\exists\, e \in G$ s.t. $eg = ge = g$, $\forall g \in G$

(c) $\forall g \in G$ $\exists g^{-1}$ s.t. $g^{-1} g = g^{-1} g = e$

subgroup



H的群乘法与G相同.



μ is a homeomorphism

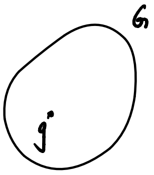
$\mu: G \rightarrow G'$

$\mu(g_1)\mu(g_2) = \mu(g_1 g_2)$

if μ is one-to-one and onto → isomorphism

特别地，对于 $\mu: G \rightarrow G$ automorphism

adjoint isomorphism



$I_g : G \rightarrow G$

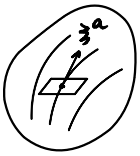
$I_g(h) := g h g^{-1}$, $\forall h \in G$

Fluid Flow （本节使用 +,-,-,- 号差）

4-velocity : ξ^a

与 ξ^a 垂直的类空超平面上，有

induced metric : h_{ab}



流体的

rotation field : $\omega_{ab} = h_{[a}{}^{m} h_{b]}{}^{n} \nabla_m \xi_n \rightarrow$ 投影的反对称部分

expansion field : $\theta_{ab} = h_{(a}{}^{m} h_{b)}{}^{n} \nabla_m \xi_n \rightarrow$ 投影的对称部分 .

$\therefore h_a{}^m h_b{}^n \nabla_m \xi_n = \omega_{ab} + \theta_{ab}$

$\downarrow$

$$(g_a{}^m - \xi_a \xi^m)(g_b{}^n - \xi_b \xi^n) \nabla_m \xi_n$$

$$= \nabla_a \xi_b - \xi_b \xi^n \cancel{\nabla_a} \xi_n - \xi_a \xi^m \nabla_m \xi_b$$

$$+ \xi_a \xi_b \xi^m \xi^n \cancel{\nabla_m} \xi_n \quad (\text{with } \nabla_m(\tfrac{1}{2}\xi^b \xi_b) = 0$$

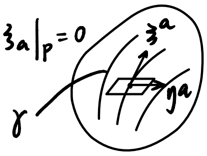
$$\Downarrow$$

$$= \nabla_a \xi_b - \xi_a \xi^m \nabla_m \xi_b \qquad \xi^b \nabla_m \xi_b = 0)$$

$\Rightarrow \nabla_a \xi_b = \theta_{ab} + \omega_{ab} + \xi_a \xi^m \nabla_m \xi_b$

对一个在 γ 上的矢量场 η^a，在 p 点有 $\eta^a \xi_a |_p = 0$

$\mathcal{L}_\xi \eta^a = 0$



即 $\xi^b \nabla_b \eta^a - \eta^b \nabla_b \xi^a = 0$

$\Rightarrow \xi^b \nabla_b \eta^a = \eta^b \nabla_b \xi^a = \eta^b(\theta_b{}^a + \omega_b{}^a + \xi_b \xi^m \nabla_m \xi^a)$

所以，在 p 点，$(\xi^b \eta_b = 0)$

有 $\xi^b \nabla_b \eta^a |_p = (\theta_b{}^a + \omega_b{}^a) \eta^b$

① $\omega_b{}^a \eta^b \perp \eta_a \quad (\omega_{ba} \eta^a \eta^b = 0)$

② $\xi^a \nabla_a (-\eta^b \eta_b) = -2 \eta^b \xi^a \nabla_a \eta_b$

$$= -2 \eta^b \eta^c (\theta_{cb} + \cancel{\omega}_{cb} + \xi_c \xi^m \nabla_m \xi_b)$$

$$= -2 \theta_{ab} \eta^a \eta^b - 2 \eta^a \xi_a \eta^b \xi^m \nabla_m \xi_b$$

Fermi 的定义:

$$\frac{D_F \eta^a}{d\tau} = \xi^b \nabla_b \eta^a + 2 A^{[a} \xi^{b]} \eta_b$$

rotation :

$$\frac{D \eta^a}{d\tau} = -\Omega^{ab} \eta_b$$

$$\tilde{\Omega}^{ab} = 2 A^{[a} \xi^{b]}$$

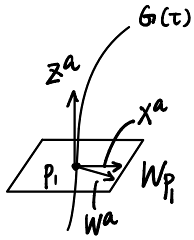
$$\hat{\Omega}^{ab} = \Omega^{ab} - \tilde{\Omega}^{ab}$$

对于 $\mathcal{L}_\xi \eta^a = 0$ ，在 p

$\xi^b \nabla_b \eta^a = \eta^b (\theta_b{}^a + \omega_b{}^a)$

所以对应有 $\theta_b{}^a + \omega_b{}^a |_p = -\Omega_b{}^a$

Fermi transport & non-rotating observers



Z^a, four-velocity

X^a 用来表示不转动的方向.

不能用平移来表示 $G(\tau)$ 上的 X^a，因为 X^a 平移可能变为非空间矢量

$$X^a Z_a = 0$$

$$Z^b \nabla_b (X^a Z_a) = 0$$ 则处处为 0

如果 X^a 是平移的，记为 X'^a

$$Z^b \nabla_b (X'^a Z_a) = X'^a Z^b \nabla_b Z_a = X'^a \underset{\substack{\downarrow \\ \text{four-acceleration}}}{A_a}$$

$$\frac{DX^a}{d\tau} \equiv Z^b \nabla_b X^a$$

$\frac{D_F}{d\tau}$, Fermi derivative operator

$\frac{D_F}{d\tau}: \mathscr{F}_G(k,l) \to \mathscr{F}_G(k,l)$ (对 geodesic, $\frac{D_F}{d\tau} = \frac{D}{d\tau}$)

(a) linear

(b) satisfy Lebnize rule

(c) 与 contraction 可交换

(d) $\frac{D_F}{d\tau} f = \frac{df}{d\tau}$, $\forall f \in \mathscr{F}_G(0,0)$

$$\frac{D_F}{d\tau} v^a = \frac{Dv^a}{d\tau} + 2A^{[a}Z^{b]} v_b$$

$$\frac{D_F w_a}{d\tau} = \frac{Dw_a}{d\tau} + (A_a \wedge Z^b) w_b$$

$$\frac{D_F v^a w_a}{d\tau} \checkmark$$
$$= w_a \frac{Dv^a}{d\tau} + w_a 2A^{[a}Z^{b]}v_b \; 0$$
$$0 + v^a \frac{Dw_a}{d\tau} + v^a 2A_{[a}Z^{b]}w_b$$

$$\frac{D_F}{d\tau} T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} = \frac{D}{d\tau} T^{a_1 \cdots a_k}{}_{b_1 \cdots b_l} + (A^{a_1} \wedge Z_{c_1}) T^{c_1 a_2 \cdots a_k}{}_{b_1 \cdots b_l}$$
$$+ \cdots + (A_{b_1} \wedge Z^{d_1}) T^{a_1 \cdots a_k}{}_{d_1 b_2 \cdots b_l} - \cdots$$

Claim: $\frac{D_F Z^a}{d\tau} = 0$

Pf: $\frac{D_F Z^a}{d\tau} = \frac{DZ^a}{d\tau} + (A^a \wedge Z^b) Z_b$

$$= \frac{DZ^a}{d\tau} - A^a = 0 \qquad □$$

$(Z^b Z^b = -1,\ A^b Z_b = 0)$

$$\downarrow$$
$$= (Z^c \nabla_c Z^b) Z_b$$
$$= \underset{\substack{\| \\ 0}}{Z^c \nabla_c (-1)} - (Z_c \nabla_c Z_b) Z^b$$

Claim: $\frac{D_F W^a}{d\tau}$ is spatial, for $\forall\, w^a$ which is spatial

or $\frac{D_F W^a}{d\tau} = h^a{}_b \frac{DW^b}{d\tau}$

Pf. $Z^a W_a = 0$

$$\begin{cases} h^a{}_b \frac{DW^b}{d\tau} W_a = \frac{DW^b}{d\tau} W_b \\ h^a{}_b \frac{DW^b}{d\tau} Z_a = 0 \end{cases}$$

(i) $\frac{D_F W^a}{d\tau} W_a = \frac{DW^a}{d\tau} W_a + (A^a \wedge Z^b) W_b W_a$

$$= \frac{DW^a}{d\tau} W_a$$

(ii) $\frac{D_F W^a}{d\tau} Z_a = \frac{DW^a}{d\tau} Z_a + (A^a \wedge Z^b) W_b Z_a$

$$= Z_a Z^b \nabla_b W^a + \underset{\searrow 0}{A^a Z_a} Z^b W_b - A^b W_b Z^a Z_a$$

$$= Z_a Z^b \nabla_b W^a + A^b W_b$$

$$\downarrow$$

$$\underset{\substack{\| \\ 0}}{Z^b \nabla_b Z_a W^a} - \underset{\substack{\| \\ -W^a A_a}}{W^a Z^b \nabla_b Z_a} \qquad □$$

Claim: $\frac{D_F}{d\tau} g_{ab} = 0$

Pf. $\frac{D_F}{d\tau} g_{ab} = \frac{D}{d\tau} g_{ab} - \underset{\substack{\downarrow \\ A_b \wedge B_a}}{(A^c \wedge B_a)} g_{cb} - \underset{\substack{\downarrow \\ A_a \wedge B_b}}{(A^c \wedge B_b)} g_{ac}$

$$= 0 \qquad □$$

rotating

$$\frac{Dv^a}{d\tau} = -\Omega^{ab} v_b \qquad \text{space-time rotation}$$

space-time rotation preserves inner product

$$\tilde{\Omega}^{ab} = A^a \wedge Z^b \ , \quad \frac{DZ^a}{d\tau} = -\tilde{\Omega}^{ab} Z_b \qquad \text{boost}$$

$$\{\underset{\substack{\downarrow \\ \text{boost}}}{\Omega_{0i}}, \underset{\substack{\downarrow \\ \text{spatial rotation}}}{\Omega_{ij}}\}$$

w^a, a spatial vector field on the world line

$$\frac{D_F w^a}{d\tau} = 0 = \frac{Dw^a}{d\tau} + (A^a \wedge Z^b) w_b \Rightarrow \frac{Dw^a}{d\tau} = -\tilde{\Omega}^{ab} w_b$$

Claim: $\frac{Dw^a}{d\tau} = -\Omega^{ab} w_b$, $\hat{\Omega}_{ab} = \Omega_{ab} - \tilde{\Omega}_{ab}$

$\hat{\Omega}_{ab}$ is a pure spatial rotation

Pf: $-\hat{\Omega}^{ab} w_b = -\Omega^{ab} w_b + \tilde{\Omega}^{ab} w_b = \frac{D_F w^a}{d\tau}$

选一组 orthonormal basis $(e_0)^a = Z^a$, $(e_1)^a = \alpha w^a$

规一化

$(-\hat{\Omega}^{ab} w_b) Z_a = 0 \Rightarrow -\hat{\Omega}^{0i} w_i = 0$, $\forall$ spatial w^a

$\therefore \hat{\Omega}^{0i} = 0 \qquad \therefore \hat{\Omega}^{ab}$ is spatial □

Claim: w^a 无 spatial rotation 的 sufficient and necessary condition 是 $\frac{D_F w^a}{d\tau} = 0$

Pf: $-\hat{\Omega}^{ab} w_b = \frac{D_F w^a}{d\tau}$, $\hat{\Omega}^{ab} = 0$ □

spatial rotation 用 spatial vector (angular velocity) 表示

$$\frac{D_F w_a}{d\tau} = -\hat{\Omega}_{ab} w^b = -\underbrace{Z^b \varepsilon_{bacd}}_{\text{induced 3 volume element}} \omega^d w^c$$

Claim: $(e_0)^a = Z^a$, $\{(e_\mu)^a\}$ 为 orthonormal basis,

任意 $G(\tau)$ 上, 空间基矢的转动角速度都一样

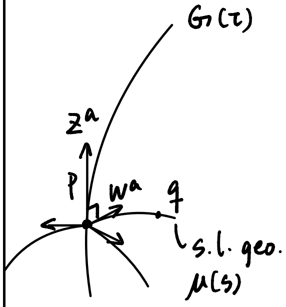
Pf: $\frac{D_F (e_i)^a}{d\tau} = -\hat{\Omega}_i^{ab} (e_i)_b$

$$\frac{D_F (e_i)^a (e_j)_a}{d\tau} = 0 = -(e_j)_a \hat{\Omega}_i^{ab} (e_i)_b - (e_i)_a \hat{\Omega}_j^{ab} (e_j)_b = 0$$

$\Rightarrow (\hat{\Omega}_i)_{ji} + (\hat{\Omega}_j)_{ij} = 0$

$\Rightarrow (\hat{\Omega}_i)_{ij} = (\hat{\Omega}_j)_{ij}$ □

Proper coordinate system



1. $A^a \neq 0$ 2. $w^a \neq 0$

q 与 $G(\tau)$ 足够近, 则

q 与 p 可以唯一地对应

$w^a = T^a|_p$

$w^i = w^a (e^i)_a$

q 点坐标:

$t(q) := \tau(p)$

$x^i(q) := s|_q w^i$

Claim: proper coordinate 的 coordinate basis 在 $G(\tau)$ 上与原本的 orthonormal basis 重合

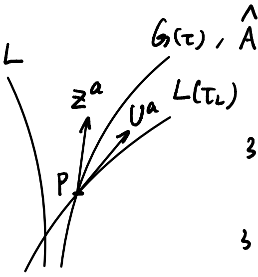
Pf: $\left(\frac{\partial}{\partial t}\right)^a\Big|_p = \left(\frac{\partial}{\partial \tau}\right)^a\Big|_p = Z^a\Big|_p = (e_0)^a\Big|_p$

w^a 方向的偏导 ←

$w^i \left(\frac{\partial}{\partial x^i}\right)^a\Big|_p = \left(\frac{\partial}{\partial s}\right)^a\Big|_p = T^a\Big|_p = w^a$

$\therefore \left(\frac{\partial}{\partial x^i}\right)^a\Big|_p = (e_i)^a\Big|_p$ □

另外 $g_{\mu\nu}|_p = \eta_{\mu\nu}$



3- velocity $u^a := \left(\frac{\partial}{\partial x^i}\right)^a \frac{dx^i(t)}{dt}$

3- acceleration $a^a := \left(\frac{\partial}{\partial x^i}\right)^a \frac{d^2 x^i(t)}{dt^2}$

Claim: $u^a = \frac{h^a{}_b U^b}{\gamma}$, $\gamma \equiv - Z_a U^a$

Pf. 选用 orthonormal basis, $(e_0)^a = Z^a$

$$h^a{}_b U^b (e^i)_a = U^i = \left(\frac{\partial}{\partial \tau_L}\right)^a (dx^i)_a \Big|_p = \frac{dx^i}{d\tau_L}\Big|_p$$

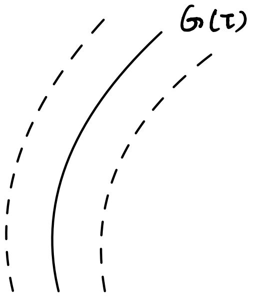
$$\gamma = -U^0 = -\left(\frac{\partial}{\partial \tau_L}\right)^a (dx^0)_a = - \frac{d\tau}{d\tau_L}\Big|_p$$

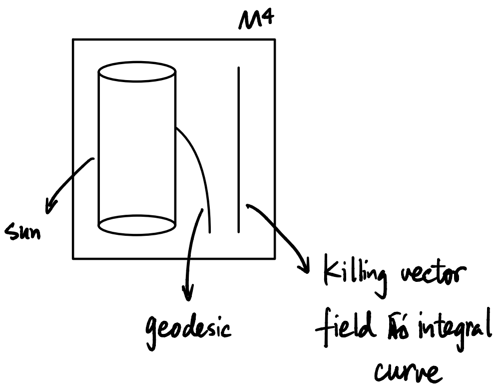
在p点 $d\tau = dt$ $\quad \therefore \gamma = - \frac{dt}{d\tau_L}\Big|_p$

$\therefore u^i = \frac{U^i}{\gamma} = \frac{dx^i}{dt}$ □

Claim: $a^a = - \hat{A}^a - 2 \varepsilon^a{}_{bc} \omega^b u^c + 2(\hat{A}_b u^b) u^a$

Equivalence priciple & local inertial frames



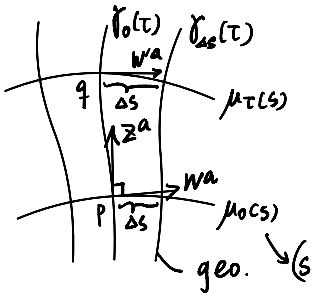


一电荷沿 killing vector field 的 integral cure 运动

才不会发出电磁波

而沿 geodesic 运动才会发出.

Geodesic deviation equation



(s 不是距离长度)

$\{\tau, s, x^2, x^3\}$

$Z^a = \left(\frac{\partial}{\partial \tau}\right)^a$

$W^a = \left(\frac{\partial}{\partial s}\right)^a$ separation vector

是两个坐标基矢

有 $0 = [Z, W]^a = Z^b \nabla_b W^a - W^b \nabla_b Z^a$

有 $Z^b \nabla_b (Z^a W_a)$

$= Z^b Z^a \nabla_b W_a$ $\quad$ 代入 (geo.)

$= \underset{\sim}{Z^a} W^b \underset{\sim\sim}{\nabla_b Z_a}$

$= \frac{1}{2} W^b \nabla_b (Z^a Z_a) = 0$

∴ 处处正交

$W^a = \left(\frac{\partial}{\partial s}\right)^a \Big|_p$

$W'^a = \left(\frac{\partial}{\partial s}\right)^a \Big|_q$

$\Rightarrow \quad W^a \Delta s = \lambda^a, \quad W'^a \Delta s = \lambda'^a$

相对速度 $\tilde{u}^a := Z^b \nabla_b \lambda^a$ is spatial

$$\left(Z_a u^a = Z_a Z^b \nabla_b \lambda^a = Z^b \nabla_b (\underset{\downarrow\atop 0}{Z_a \lambda^a}) - \lambda^a \underbrace{Z^b \nabla_b Z_a}_{=0} = 0 \right)$$

相对加速度 $\tilde{a}^a := Z^b \nabla_b \tilde{u}^a$

将 Δs 提出, 定义

$u^a := Z^b \nabla_b W^a \quad , \quad a^a := Z^b \nabla_b u^a$

Claim: $a^c = - R_{abd}{}^c Z^a W^b Z^d$

Pf: $a^c = Z^a \nabla_a (Z^b \nabla_b W^c) = Z^a \nabla_a (W^b \nabla_b Z^c)$

$\left([Z, W]^a = 0 = Z^b \nabla_b W^a - W^b \nabla_b Z^a \right)$

$$= \underbrace{Z^a W^b \nabla_a \nabla_b Z^c}_{p^c} + \underbrace{Z^a (\nabla_b Z^c) \nabla_a W^b}_{q^c}$$

$$p^c = Z^a W^b \nabla_a \nabla_b Z^c = Z^a W^b \nabla_b \nabla_a Z^c - Z^a W^b R_{abd}{}^c Z^d$$

$$= W^b \nabla_b \underset{\substack{\| \\ 0}}{\left(Z^a \nabla_a Z^c\right)} - \underbrace{W^b (\nabla_a Z^c) \nabla_b Z^a}_{\downarrow \text{ 代入 } W^b \nabla_b Z^a = Z^b \nabla_b W^a} - Z^a W^b R_{abd}{}^c Z^d$$

$$= -(\nabla_a Z^c) Z^b \nabla_b W^a = -q^c$$

$$a^c = p^c + q^c = -R_{abd}{}^c Z^a W^b Z^d$$

□

$\tilde{u}^b|_{\tau=0} = 0$ 初始平行

The Einstein field equation

$$R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab}$$

Linear approximation & the Newtonian gravitation theory

$$(\mathbb{R}^4, g_{ab}) \qquad g_{ab} = \eta_{ab} + \gamma_{ab}$$

$$\downarrow$$

$$\{t, x^i\} \quad |\gamma_{\mu\nu}| \ll 1$$

$$g^{ab} = \eta^{ab} - \gamma^{ab} \quad (\gamma^{ab} = \eta^{ac}\eta^{bd}\gamma_{cd})$$

除此之外，都用 η^{ab} 升降指标

$$(g^{ab} g_{bc} = (\eta^{ab} - \gamma^{ab})(\eta_{bc} + \gamma_{bc}) = \delta^a{}_c - \gamma^{ab}\gamma_{bc})$$

$$R_{abc}{}^d = -2\partial_{[a}\Gamma^d{}_{b]c} + 2\underset{\substack{\downarrow \\ \text{second order}}}{\Gamma^e{}_{c[a}\Gamma^d{}_{b]e}}$$

$$\Gamma^c{}_{ab} = \frac{1}{2} g^{cd}(2g_{d(a,b)} - g_{ab,d})$$

$$\text{近似} = \frac{1}{2}\eta^{cd}(2\gamma_{d(a,b)} - \gamma_{ab,d})$$

$$= \partial_{(a}\gamma^c{}_{b)} - \frac{1}{2}\partial^c\gamma_{ab}$$

代入

$$R_{abc}{}^d = -(\underset{\substack{\downarrow \\ 0}}{\partial_{[a}\partial_{b]}}\gamma^d{}_c + \partial_c\partial_{[a}\gamma^d{}_{b]} - \partial^d\partial_{[a}\gamma_{b]c})$$

$$= \partial^d\partial_{[a}\gamma_{b]c} - \partial_c\partial_{[a}\gamma^d{}_{b]}$$

$$R_{ac} = R_{abc}{}^b = \underset{\substack{\downarrow \\ \frac{1}{2}(\partial_b\partial_a\gamma^b{}_c - \partial_b\partial^b\gamma_{ac})}}{\partial^b\partial_{[a}\gamma_{b]c}} - \frac{1}{2}(\partial_c\partial_a\gamma^b{}_b - \partial_c\partial_b\gamma^b{}_a)$$

$$= \partial_b\partial_{(a}\gamma^b{}_{c)} - \frac{1}{2}(\partial_b\partial^b\gamma_{ac} + \partial_a\partial_c\gamma^b{}_b)$$

$$R = R_a{}^a = \partial_a\partial_b\gamma^{ab} - \partial^b\partial_b\gamma^a{}_a$$

$$G_{ab} = R_{ab} - \frac{1}{2}g_{ab}R \overset{\text{近似}}{=} R_{ab} - \frac{1}{2}\eta_{ab}R$$

$$= \partial_c\partial_{(a}\gamma^c{}_{b)} - \frac{1}{2}(\partial_c\partial^c\gamma_{ab} + \partial_a\partial_b\gamma)$$

$$- \frac{1}{2}\eta_{ab}(\partial_c\partial_d\gamma^{cd} - \partial^c\partial_c\gamma) \qquad (\text{记 } \gamma^a{}_a = \gamma)$$

令 $\bar{\gamma}_{ab} = \gamma_{ab} - \frac{1}{2}\eta_{ab}\gamma$ 代入

$$G_{ab} = \partial_c\partial_{(a}\bar{\gamma}^c{}_{b)} + \underset{\substack{\downarrow \\ \frac{1}{2}\partial_{(a}\partial_{b)}\gamma = \frac{1}{2}\partial_{\cancel{a}}\partial_b\gamma}}{\partial_c\partial_{(a}\frac{1}{2}\eta^c{}_{b)}\gamma}$$

$$\left.\begin{aligned} &- \frac{1}{2}\partial_c\partial^c\bar{\gamma}_{ab} - \frac{1}{4}\partial_c\cancel{\partial^c}(\eta_{ab}\gamma) \\ &- \frac{1}{2}\eta_{ab}\partial_c\partial_d\bar{\gamma}^{cd} - \frac{1}{4}\eta_{ab}\cancel{\partial_c}\partial^c\gamma \end{aligned}\right\}$$

$$- \frac{1}{2}\partial_{\cancel{a}}\partial_b\gamma + \frac{1}{2}\eta_{ab}\cancel{\partial^c}\partial_c\gamma$$

$$= \partial_c\partial_{(a}\bar{\gamma}^c{}_{b)} - \frac{1}{2}\partial^c\partial_c\bar{\gamma}_{ab} - \frac{1}{2}\eta_{ab}\partial_c\partial_d\bar{\gamma}^{cd}$$

gauge transformation : $\tilde{\gamma}_{ab} = \gamma_{ab} + 2\partial_{(a}\xi_{b)}$

$$\tilde{\bar{\gamma}}_{ab} = \tilde{\gamma}_{ab} - \frac{1}{2}\eta_{ab}\tilde{\gamma}$$

$$= \gamma_{ab} + 2\partial_{(a}\xi_{b)} - \frac{1}{2}\eta_{ab}(\gamma + 2\partial^c\xi_c)$$

$$= \bar{\gamma}_{ab} + 2\partial_{(a}\xi_{b)} - \eta_{ab}\partial^c\xi_c$$

$$\tilde{R}_{abc}{}^d - R_{abc}{}^d = \partial^d\underset{(=0)}{\partial_{[a}\partial_{b]}}\xi_c + \partial^d\partial_c\cancel{\partial_{[a}}\xi_{b]}$$

$$- \partial_c\underset{(=0)}{\partial_{[a}\partial_{b]}}\xi^d - \partial_c\cancel{\partial^d}\partial_{[a}\xi_{b]}$$

$$= 0$$

选择 gauge tran. $\partial^b \tilde{\bar{\gamma}}_{ab} = 0$

对于任意不满足此条件的 γ_{ab}

$$\partial^b(\gamma_{ab} + 2\partial_{(a}\xi_{b)}) - \partial^b(\frac{1}{2}\eta_{ab}(\gamma^c{}_c + 2\partial^c\xi_c)) = 0$$

$$\partial^b\gamma_{ab} - \frac{1}{2}\partial_a\gamma^c{}_c + 2\partial^b\partial_{(a}\xi_{b)} - \partial_a\cancel{\partial^c}\xi_c = 0$$

$$\downarrow$$

$$\partial^b\cancel{\partial_a}\xi_b + \partial^b\partial_b\xi_a$$

$\partial^b\gamma_{ab} - \frac{1}{2}\partial_a\gamma + \partial^b\partial_b\xi_a = 0$ 解一定存在

找到满足此条件的 $\tilde{\gamma}_{ab}$

$$\tilde{G}_{ab} = -\frac{1}{2}\partial^c\partial_c\tilde{\bar{\gamma}}_{ab}$$

field eq. simplified :

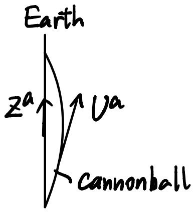
$$-\frac{1}{2}\partial^c\partial_c\tilde{\bar{\gamma}}_{ab} = 8\pi T_{ab} \Rightarrow \partial^c\partial_c\tilde{\bar{\gamma}}_{ab} = -16\pi T_{ab}$$

Newtonian :

(1) $T_{ab} = \rho(dx^0)_a(dx^0)_b$ ($T_{00} \cong \rho$, $T_{i0} \cong 0$, $T_{ij} \cong 0$)

(2) (a) $\partial\bar{\gamma}_{\mu\nu}/\partial t \cong 0$

(b) $U^a \cong (\frac{\partial}{\partial t})^a \equiv Z^a$



$$\partial^\sigma\partial_\sigma\bar{\gamma}_{\mu\nu} = \partial^i\partial_i\bar{\gamma}_{\mu\nu} \quad (\partial_0\gamma_{\mu\nu} \cong 0)$$
$$= \nabla^2\bar{\gamma}_{\mu\nu}$$

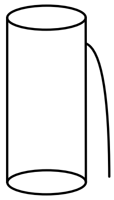
$$\left.\begin{cases} \nabla^2\bar{\gamma}_{00} = -16\pi\rho, \\ \nabla^2\bar{\gamma}_{0i} = 0, \\ \nabla^2\bar{\gamma}_{ij} = 0, \end{cases}\right\} \Rightarrow \bar{\gamma}_{0i} = 0,\ \bar{\gamma}_{ij} = 0$$

$$\bar{\gamma}^\mu{}_\mu = \gamma - \frac{1}{2}\eta^\mu{}_\mu\gamma = -\gamma \Rightarrow \gamma = -\bar{\gamma}^\mu{}_\mu = -\bar{\gamma}^0{}_0 = \bar{\gamma}_{00}$$

$$\gamma_{\mu\nu} = \bar{\gamma}_{\mu\nu} + \frac{1}{2}\eta_{\mu\nu}\gamma \Rightarrow \gamma_{ij} = \frac{1}{2}\eta_{ij}\bar{\gamma}_{00},\ \gamma_{0i} = 0,\ \gamma_{00} = \frac{1}{2}\bar{\gamma}_{00}$$

$\bar{\gamma}_{00} \equiv -4\phi$ 则，得到 $\nabla^2\phi = 4\pi\rho$

free falling



$$\frac{d^2u^\mu}{d\tau^2} + \Gamma^\mu{}_{\nu\sigma}u^\nu u^\sigma = 0$$

$$\frac{d^2x^0}{dt^2} = -\Gamma^0{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} - 2\Gamma^0{}_{0i}\frac{dx^0}{dt}\frac{dx^i}{dt}$$

$$\downarrow$$

$$0 \qquad -\Gamma^0{}_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} \qquad (\frac{dx^i}{dt} = 0) \qquad x^0 = t$$

$$\searrow 1$$

$$\Rightarrow 0 = -\Gamma^0{}_{ij}$$

$$\frac{d^2x^i}{dt^2} = -\Gamma^i{}_{00}\frac{dx^0}{dt}\frac{dx^0}{dt} + 0 = -\Gamma^i{}_{00}$$

$$\Gamma^\rho{}_{\mu\nu} = \frac{1}{2}\eta^{\rho\sigma}(\gamma_{\sigma\mu,\nu} + \gamma_{\sigma\nu,\mu} - \gamma_{\mu\nu,\sigma})$$

$$\Gamma^0{}_{00} = -\frac{1}{2}\gamma_{00,0} \cong 0$$

$$\Gamma^0{}_{i0} = -\frac{1}{2}(\gamma_{0i,0} + \gamma_{00,i} - \gamma_{0i,0}) = -\frac{1}{2}\gamma_{00,i}$$

$$\Gamma^i{}_{00} = \frac{1}{2}\eta^{i\mu}(2\gamma_{\mu0,0} - \gamma_{00,\mu}) \cong -\frac{1}{2}\partial^i\gamma_{00}$$

$$\frac{d^2x^i}{dt^2} = \frac{1}{2}\partial^i\gamma_{00} = \frac{1}{2}\partial^i(\frac{1}{2}(-4\phi)) = -\partial^i\phi$$

Cosmology

Kinetic of the Universe

Cosmological principle

(1) spatially, homogéneous 空间均匀

spatially isotropic 各向同性 $\downarrow$

surface of homogeneity

Possible Spatial Geometries of the Universe

1. (M, g_{ab}) is spatially homogeneous if.

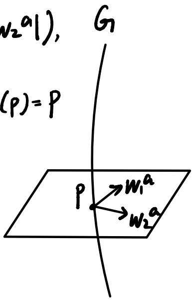
$\exists\ \{\Sigma_t\}$, s.t. $\forall t \in \mathbb{R}$, $p, q \in \Sigma_t$,

$\exists$ diff $\phi: \Sigma_t \to \Sigma_t$ s.t. ① $\phi^* h_{ab} = h_{ab}$. ② $\phi(p) = q$

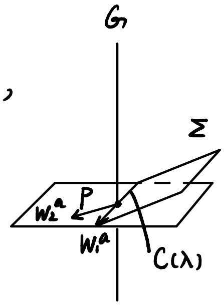
2. $\mathscr{R}$, $\forall p \in G$, $w_1{}^a, w_2{}^a \in W_p$ $(|w_1{}^a| = |w_2{}^a|)$,

$\exists$ diff $\phi: M \to M$ s.t. ① $\psi^* g_{ab} = g_{ab}$ ② $\psi(p) = p$

③ $\psi_* w_1{}^a = w_2{}^a$ (isotropic)



Claim: $\{\Sigma\}$ is unique $\Rightarrow \Sigma \perp G$



Pf. ① $\psi^* g_{ab} = g_{ab}$

$\Rightarrow \psi[\Sigma]$ is a surface of homo.$^{(1)}$

② $\psi(p) = p \Rightarrow p \in \psi[\Sigma]$

$\Rightarrow p \in (\Sigma \cap \psi[\Sigma])$

由于 $\{\Sigma\}$ is unique $\Rightarrow \psi[\Sigma] = \Sigma$

③ $w_1{}^a = \left(\frac{\partial}{\partial\lambda}\right)^a \Rightarrow \psi^* w_1{}^a$ is the tangent vector$^{(2)}$ of $\psi(c(\lambda))$，而 $\psi(c(\lambda)) \in \Sigma$

$\therefore \psi^* w_1{}^a \in \Sigma$ 而取 $w_2{}^a \notin \Sigma$ 则 ③不成立

(1) Pf:

对于 Σ, $\exists$ diff $\phi: \Sigma \to \Sigma$ s.t. ① $\phi^* h_{ab} = h_{ab}$, ② $\phi(p) = q$

对于 $\psi[\Sigma]$, $\exists$ diff $\psi \circ \phi \circ \psi^{-1}: \psi[\Sigma] \to \psi[\Sigma]$

且 ① $\psi^* \circ \phi^* \circ \psi^{*-1} h'_{ab} = h'_{ab}$, ② $\psi \circ \phi \circ \psi^{-1}(p') = q'$

(2) Pf:

$\psi_* T^a(f) = T^a(\psi f) = \frac{\partial}{\partial\lambda} f(\psi(c(\lambda)))$

$\psi_* T^a$ is tangent to $\psi(c(\lambda))$ □

Space of constant curvature

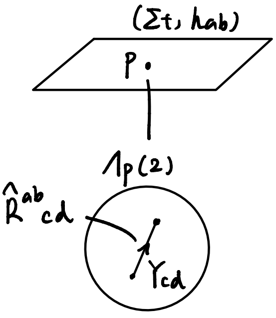
(M, g_{ab}) $R_{abcd} = K(g_{ac} g_{bd} - g_{bc} g_{ad}) = 2K g_{c[a} g_{b]d}$

① highest symmetry $\dim \mathscr{K} = \frac{n(n+1)}{2}$

② 空间维数、度规号差(signature)给定，空间的局域性质由K决定(证明较难)

Claim: $\hat{R}_{abcd} = 2K h_{c[a} h_{b]d}$, (Σ_t, h_{ab})

Pf. $\dim \Lambda_p(2) = \frac{3!}{2!(3-2)!} = 3$



$\hat{R}_{ab}{}^{cd} Y_{cd} \in \Lambda_p(2)$,

$\forall Y_{cd} \in \Lambda_p(2)$

$\therefore \hat{R}_{ab}{}^{cd}: \Lambda_p(2) \to \Lambda_p(2)$

$\hat{R}_{ab}{}^{cd} \mapsto L\ (3\times3)$

$\hat{R}_{abcd} = \hat{R}_{cdab} \Rightarrow L$ is symmetric $\overset{\text{basis}}{\Rightarrow} L = \begin{pmatrix} L_{11} & & \\ & L_{22} & \\ & & L_{33} \end{pmatrix}$

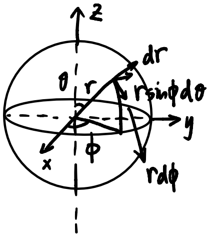
isotropic & $\{\Sigma_t\}$ unique $\Rightarrow L = 2KI$

I 对应于 $\delta_a{}^{[c} \delta^{d]}{}_b$ □

(1) $K = 0$ $\quad dl^2 = dx^2 + dy^2 + dz^2$

$= d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2)$ $\quad \begin{cases} x = \psi \sin\theta \cos\phi \\ y = \psi \sin\theta \sin\phi \\ z = \psi \cos\theta \end{cases}$

(2) $K > 0$



$x = r\sin\theta \cos\phi$

$y = r\sin\theta \sin\phi$

$z = r\cos\theta$

$\Downarrow$ 4-dim

$x = r\sin\psi \sin\theta \cos\phi$

$y = r\sin\psi \sin\theta \sin\phi$

$z = r\sin\psi \cos\theta$

$w = r\cos\psi$

$dl^2 = r^2(d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2))$

$\Rightarrow \hat{R}^{ab}{}_{cd} = \frac{2}{r^2} \delta_a{}^{[c} \delta_b{}^{d]}$ $\quad K = \frac{1}{r^2} > 0$

(3) $K < 0$ $\quad t^2 - x^2 - y^2 - z^2 = \xi^2$ (3维双曲面)

用闵氏度规 $ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$

$$\{t, x, y, z\} \longmapsto \{\xi, \psi, \theta, \phi\}$$

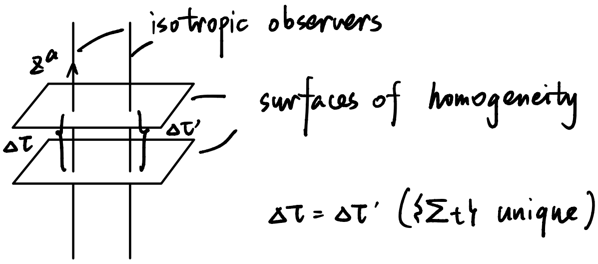
$$x = \xi \,\mathrm{sh}\psi \sin\theta \cos\phi$$
$$y = \xi \,\mathrm{sh}\psi \sin\theta \sin\phi$$
$$z = \xi \,\mathrm{sh}\psi \cos\theta$$
$$t = \xi \,\mathrm{ch}\psi$$

$$ds^2 = -d\xi^2 + \underbrace{\xi^2[d\psi^2 + \mathrm{sh}^2\psi(d\theta^2 + \sin^2\theta\, d\varphi^2)]}_{dl^2}$$

$$\hat{R}_{ab}{}^{cd} = -\frac{2}{\xi^2}\delta_a{}^{[c}\delta_b{}^{d]} \qquad k = -\frac{1}{\xi^2} < 0$$

The Robertson-Walker metric

$\{t, x^i\}$, $ds^2 = g_{00}dt^2 + 2g_{0i}dt\,dx^i + g_{ij}dx^i dx^j$



$\Delta\tau = \Delta\tau'$ ($\{\Sigma_t\}$ unique)

$x^1 \equiv \psi$, $x^2 \equiv \theta$, $x^3 = \phi$, $t \equiv \tau$ $\quad (\frac{\partial}{\partial t})^a = (\frac{\partial}{\partial \tau})^a \equiv Z^a$

$\Rightarrow g_{00} = g_{ab}Z^aZ^b = -1$, $g_{0i} = 0$, $g_{ij} = h_{ij}$

h_{ij} 是 t与x 的函数 $h_{ij}(t, x)$

而 h_{ij} 可以分离变量 $h_{ij}(t, x) = a^2(t)\hat{h}_{ij}(x)$

Pf: P_{370} 选读 (很麻烦)

$$ds^2 = -dt^2 + a^2(t)\underbrace{\hat{h}_{ij}(x)\, dx^i dx^j}_{dl^2}$$

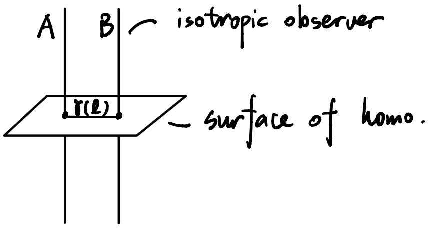
(a) $ds^2 = -dt^2 + a^2(t)[d\psi^2 + \sin^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]$, $\hat{R}_{ab}{}^{cd} = \frac{2}{a^2(t)}\delta_a{}^{[c}\delta_b{}^{d]}$

(b) $ds^2 = -dt^2 + a^2(t)[d\psi^2 + \psi^2(d\theta^2 + \sin^2\theta\, d\phi^2)]$, $\hat{R}_{ab}{}^{cd} = 0$

(c) $ds^2 = -dt^2 + a^2(t)[d\psi^2 + \mathrm{sh}^2\psi(d\theta^2 + \sin^2\theta\, d\phi^2)]$, $\hat{R}_{ab}{}^{cd} = -\frac{2}{a^2(t)}\delta_a{}^{[c}\delta_b{}^{d]}$

$$ds^2 = -dt^2 + a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\right] \quad k = \begin{cases} 1, & k>0 \\ 0, & k=0 \\ -1, & k<0 \end{cases} \quad \left(r = \begin{cases} \sin\psi \\ \psi \\ \mathrm{sh}\psi \end{cases}\right)$$

Scale factor of the universe



3维 + positive definite metric

且 metric 并不复杂

A.B 之间有最短的一条 geodesic

$$D_{AB} = \int_A^B \sqrt{h_{ab}(\frac{\partial}{\partial l})^a(\frac{\partial}{\partial l})^b}\, dl$$

$$= a(t)\underbrace{\int_A^B \sqrt{\hat{h}_{ab}(\frac{\partial}{\partial l})^a(\frac{\partial}{\partial l})^b}\, dl}_{\hat{D}_{AB}} = a(t)\hat{D}_{AB}$$

对于 $K > 0$

$$V = \int \hat{\underline{\varepsilon}} = \int \sqrt{h}\, d\psi \wedge d\theta \wedge d\phi$$

$$h = (a^2(t))^3 \begin{vmatrix} 1 & & \\ & \sin^2\psi & \\ & & \sin^2\psi \sin^2\theta \end{vmatrix} = a^6(t)\sin^4\psi \sin^2\theta$$

$$V = \int a^3(t)\sin^2\psi \sin\theta\, d\psi \wedge d\theta \wedge d\phi$$

$$= a^3(t)\int_0^\pi \sin^2\psi\, d\psi \int_0^\pi \sin\theta\, d\theta \int_0^{2\pi} d\phi = 2\pi^2 a^3(t)$$

$$\left(\begin{array}{l} X = r\sin\psi \\ y = r\sin\psi \\ z = r\sin\psi \end{array} \boxed{\begin{array}{l} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ \cos\theta \end{array}} \rightarrow \text{与2维球一样} \right)$$

$$W = r\cos\psi \rightarrow -1, 1 \rightarrow 0, \pi$$

Dynamics of the universe

Hubble's law

$$z = \frac{\lambda' - \lambda}{\lambda} , \quad z \propto D$$

$$z = \frac{\omega}{\omega'} - 1 \qquad \omega' = \sqrt{\frac{1-u}{1+u}}\, \omega$$

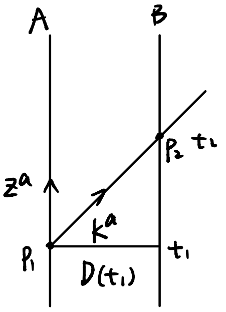
$$\Rightarrow z = \sqrt{\frac{1+u}{1-u}} - 1 \approx u$$

$$\Rightarrow u \propto D \quad 或 \quad u_0 = H_0 D_0$$

$$u(t) \equiv \frac{dD(t)}{dt} = \hat{D} \frac{da(t)}{dt} = D(t) \underset{\underset{H(t)}{|||}}{\frac{\dot{a}(t)}{a(t)}} = H(t) D(t)$$

$$(D(t) = a(t) \hat{D})$$

Cosmological redshift



$$\omega_1 = -g_{ab} Z^a K^b = -g_{ab} \left(\frac{\partial}{\partial t}\right)^a K^b$$

$$K^b \equiv \left(\frac{\partial}{\partial \beta}\right)^b = \left(\frac{\partial}{\partial t}\right)^b \frac{dt}{d\beta} + \left(\frac{\partial}{\partial x^i}\right)^b \frac{dx^i}{d\beta}$$

代入

$$\omega_1 = -g_{ab} \left(\frac{\partial}{\partial t}\right)^a \left(\frac{\partial}{\partial t}\right)^b \frac{dt}{d\beta} \quad (g_{0i} = 0)$$

$$= -g_{00} \frac{dt}{d\beta} \quad (g_{00} = -1)$$

$$= \left.\frac{dt}{d\beta}\right|_{p_1}$$

$$\omega_2 = \left.\frac{dt}{d\beta}\right|_{p_2} \quad 或 \quad \omega(\beta) = \frac{dt(\beta)}{d\beta}$$

$$0 = \frac{d^2 x^\mu}{d\beta^2} + \Gamma^\mu{}_{\nu\sigma} \frac{dx^\nu}{d\beta} \frac{dx^\sigma}{d\beta}$$

在坐标 $\{\overset{0}{t}, \overset{1}{r}, \overset{2}{\theta}, \overset{3}{\phi}\}$ 下：

$$0 = \frac{d^2\theta}{d\beta^2} + 2\frac{\dot{a}}{a} \frac{dt}{d\beta} \frac{d\theta}{d\beta} + 2\frac{1}{r} \frac{dr}{d\beta} \frac{d\theta}{d\beta} - \sin\theta \cos\theta \left(\frac{d\phi}{d\beta}\right)^2$$

$$0 = \frac{d^2\phi}{d\beta^2} + 2\frac{\dot{a}}{a} \frac{dt}{d\beta} \frac{d\phi}{d\beta} + 2\frac{1}{r} \frac{dr}{d\beta} \frac{d\phi}{d\beta} + 2\cot\theta \frac{d\theta}{d\beta} \frac{d\phi}{d\beta}$$

$$\left.\frac{d\theta}{d\beta}\right|_{p_1} = 0 , \quad \left.\frac{d\phi}{d\beta}\right|_{p_1} = 0 \quad (选初始条件)$$

由 geodesic 的方程知,

$\frac{d\theta}{d\beta}$ 与 $\frac{d\phi}{d\beta}$ 恒为 0

$$K^a = \left(\frac{\partial}{\partial t}\right)^a \frac{dt}{d\beta} + \left(\frac{\partial}{\partial r}\right)^a \frac{dr}{d\beta} + \underbrace{\left(\frac{\partial}{\partial \theta}\right)^a \frac{d\theta}{d\beta} + \left(\frac{\partial}{\partial \phi}\right)^a \frac{d\phi}{d\beta}}_{0}$$

$$0 = \frac{d^2 t}{d\beta^2} + \frac{a\dot{a}}{1-kr^2} \left(\frac{dr}{d\beta}\right)^2 + \cdots\cdots \quad (由于 \frac{d\theta}{d\beta} = \frac{d\phi}{d\beta} = 0, 后几项略)$$

$$0 = g_{ab} K^a K^b = g_{ab} \left[\left(\frac{\partial}{\partial t}\right)^a \frac{dt}{d\beta} + \left(\frac{\partial}{\partial r}\right)^a \frac{dr}{d\beta}\right]\left[\left(\frac{\partial}{\partial t}\right)^b \frac{dt}{d\beta} + \left(\frac{\partial}{\partial r}\right)^b \frac{dr}{d\beta}\right]$$

$$= -\left(\frac{dt}{d\beta}\right)^2 + \frac{a^2}{1-kr^2} \left(\frac{dr}{d\beta}\right)^2$$

$$\downarrow$$

$$\frac{d\omega}{d\beta} + \omega^2 \frac{\dot{a}}{a} = 0$$

$$\Rightarrow a \frac{d\omega}{d\beta} + \frac{da}{dt} \omega \frac{dt}{d\beta} = 0$$

$$\Rightarrow \frac{da\omega}{d\beta} = 0 \qquad \omega \propto a^{-1}, \quad \lambda \propto a$$

$$a(t_2) \cong a(t_1) + \dot{a}(t_1)(t_2 - t_1) = a(t_1) + \dot{a}(t_1) D(t_1)$$

$$(t_2 - t_1 \cong D(t_1), (0 = ds^2 = -dt^2 + dl^2))$$

$$\frac{\omega_1}{\omega_2} = \frac{a(t_2)}{a(t_1)} = 1 + \frac{\dot{a}(t_1)}{a(t_1)} D(t_1) = 1 + H(t_1) D(t_1)$$

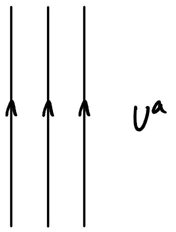
$$z = \frac{\lambda_2 - \lambda_1}{\lambda_1} = \frac{\omega_1}{\omega_2} - 1 = H(t_1) D(t_1)$$

Evolution of the scale factor

$$\text{contents} \begin{cases} \text{matter} & T_{ab}(\text{matter}) = \rho_M U_a U_b \\ \text{radiation} & T_{ab}(\text{rad}) = \rho_R U_a U_b + P(g_{ab} + U_a U_b) \\ & (P = \rho_R/3) \end{cases}$$

$$T_{ab}(\text{total}) = \rho U_a U_b + P(g_{ab} + U_a U_b)$$

$$G_{ab} = 8\pi T_{ab}$$



$$\begin{cases} T_{00} = T_{ab} U^a U^b = \rho \\ T_{i0} = T_{0i} = 0 \\ T_{ij} = P g_{ij} \end{cases}$$

$$\begin{cases} G_{00} = \frac{3(\dot{a}^2+k)}{a^2} = 8\pi\rho & ① \\ G_{ij} = -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) g_{ij} = 8\pi P g_{ij} & ② \end{cases}$$

②与①

$$\Rightarrow \frac{2\ddot{a}}{a} = -8\pi\left(P + \frac{\rho}{3}\right) \Rightarrow 3\ddot{a} = -4\pi a(\rho + 3P) \quad ③ \quad \Delta$$

$a(0) = 0$ → big bang singularity

①: $3(\dot{a}^2 + \underset{(\pm 1 或 0)}{\underset{\downarrow}{k}}) = 8\pi\rho a^2 \Rightarrow 3\dot{a}\ddot{a} = 8\pi\rho a\dot{a} + 4\pi\dot{\rho}a^2$

代入③ $-4\pi a(\rho+3P)\dot{a} = 4\pi(2\rho a\dot{a} + \dot{\rho}a^2)$

$$\Rightarrow -(\rho+3P)\dot{a} = 2\rho\dot{a} + \dot{\rho}a$$

$$\Rightarrow 3(\rho+P)\dot{a} = -\dot{\rho}a$$

$$\Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}(\rho+P) \quad \Delta$$

(不用附加物质量守恒，因为已包含 $\nabla_a T^{ab}=0$)

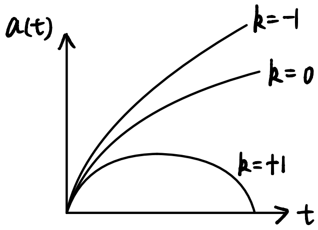
Dust Universe: $P=0 \Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}\rho \Rightarrow \rho \propto a^{-3}$

①: $\dot{a}^2 = \frac{8\pi}{3}\rho a^2 - k$ 令 $A \equiv \frac{8\pi}{3}\rho a^3 = \text{Constant}$

$\Rightarrow \dot{a}^2 = Aa^{-1} - k$ $\hat{t}(t) \equiv \int_0^t \frac{dt}{a(t)} \Rightarrow \dot{a} = \frac{da}{d\hat{t}}\frac{1}{a}$

$$\Rightarrow \frac{da}{d\hat{t}} = Aa - ka^2$$

$$\begin{cases} k=+1, & a = A(1-\cos\hat{t})/2, \ t = A(\hat{t}-\sin\hat{t})/2 \\ k=0, & a = (9A/4)^{1/3} t^{2/3} \\ k=-1, & a = A(\mathrm{ch}\hat{t}-1)/2, \ t = A(\mathrm{sh}\hat{t}-\hat{t})/2 \end{cases}$$

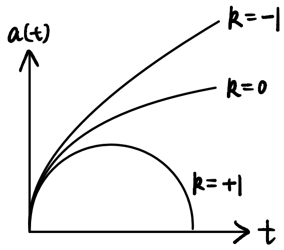


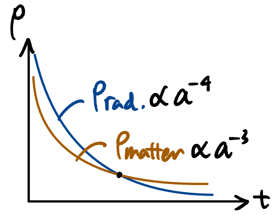
Radiation Universe: $P = \rho/3 \Rightarrow \dot{\rho} = -3\frac{\dot{a}}{a}\frac{4}{3}\rho \Rightarrow \rho \propto a^{-4}$

①: $\dot{a}^2 = \frac{8\pi}{3}\rho a^2 - k$ 令 $B^2 \equiv \frac{8\pi}{3}\rho a^4 = \text{constant}$

$$\Rightarrow \dot{a}^2 = B^2 a^{-2} - k$$

$$\Rightarrow a^2 = 2Bt - kt^2$$





早期宇宙 radiation占主导地位

$$H_0 \cong (20\ \text{km/s})/百万光年$$

$$t_0 \cong H_0^{-1} \cong 137\ 亿年$$

The cosmological constant & the Einstein's static universe

$$①: \frac{3(\dot{a}^2+k)}{a^2} = 8\pi\rho \overset{\text{static}}{\Longrightarrow} 3k = 8\pi\rho a^2$$

$$②: -\left(\frac{2\ddot{a}}{a} + \frac{\dot{a}^2+k}{a^2}\right) = 8\pi P \overset{\text{static}}{\Longrightarrow} -k = 8\pi P a^2$$

$$\left.\begin{matrix} 3k = 8\pi\rho a^2 \\ -k = 8\pi P a^2 \end{matrix}\right\} \Rightarrow \times$$

$$\tilde{G}_{ab} = 8\pi T_{ab}$$

a. $\tilde{G}_{ab} = \tilde{G}_{ba}$ b. $\nabla^a \tilde{G}_{ab} = 0$

$$\Rightarrow \tilde{G}_{ab} = G_{ab} + \Lambda g_{ab} \quad (\Lambda = \text{Constant})$$

$$G_{ab} = 8\pi\left(T_{ab} - \frac{\Lambda}{8\pi}g_{ab}\right)$$

original $\bar{T}_{ab}$, $\bar{P}$, $\bar{\rho}$

new $T_{ab} = \bar{T}_{ab} - \frac{\Lambda}{8\pi}g_{ab}$, P, ρ

$$\rho = T_{00} = \bar{T}_{00} - \frac{\Lambda}{8\pi} g_{00} = \bar{\rho} + \frac{\Lambda}{8\pi}$$

$$p g_{ij} = T_{ij} = \bar{T}_{ij} - \frac{\Lambda}{8\pi} g_{ij} = \left(\bar{p} - \frac{\Lambda}{8\pi}\right) g_{ij}$$

$$T_{0i} = \bar{T}_{0i} - \frac{\Lambda}{8\pi} g_{0i} = 0$$

对于 static universe

$$\begin{cases} 3k = 8\pi \rho a^2 = 8\pi a^2 \left(\bar{\rho} + \frac{\Lambda}{8\pi}\right) \\ -k = 8\pi p a^2 = 8\pi a^2 \left(\bar{p} - \frac{\Lambda}{8\pi}\right) \end{cases}$$

$$2k = 8\pi a^2 (\bar{\rho} + \bar{p}) \Rightarrow k = +1$$

对 dust universe $\Lambda = \frac{1}{a^2}$, $\bar{\rho} = \frac{1}{4\pi a^2}$

$$ds^2 = -dt + a^2 [d\phi^2 + \sin^2\psi (d\theta^2 + \sin^2\theta d\phi^2)]$$

The thermal history of our universe

$\rho_{Rad.} \propto T^4$, $\rho_{Rad.} \propto a^{-4}$ $\Rightarrow T \propto a^{-1}$

(radiation 占主导地位的时期)

for early universe $\dot{a}^2(t) = B^2 a^{-2} - k$, $a \ll 1$, $B^2 a^{-2} \gg k$

$\Rightarrow \dot{a}^2(t) \cong B^2 a^{-2} \Rightarrow a \cong \sqrt{2Bt}$

$\Rightarrow T \propto t^{-\frac{1}{2}}$

actually $T = \frac{10^{10}}{\sqrt{t}}$ for early universe

Quantum Mechanics

spin s
- $\frac{1}{2}, \frac{3}{2}, \cdots$ (electron, proton, neutron $\cdots$) Fermion
- $0, 1, 2 \cdots$ ($\gamma, \cdots$) Boson

Partical Physics

1. Interactions

strong , EM , weak , Gravitation

2. Antiparticles

positron ← cosmic rays

3. Neutrinos

4. Mesons

particles
- quarks — hadron ($p, n, \pi^+, \pi^- \cdots$)
- leptons
  - e, ν_e
  - μ, ν_μ
  - τ, ν_τ
- intermidiate bosons ($\gamma, g, \cdots$)

5. Hadron structure

quark : R G B

gluon

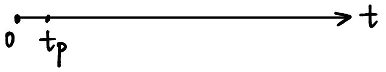
	down	up	strange	charm	bottom	top
charge (e)	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$	$-\frac{1}{3}$	$\frac{2}{3}$

$p = d\,u\,u$, $n = d\,d\,u$

$\pi^+ = \bar{d}\,u$, $\pi^- = d\,\bar{u}$

Unification of interactions

Thermal equilibrium in the early universe



$t_p = \left(\frac{\hbar G}{c^5}\right)^{\frac{1}{2}} \cong 10^{-44}$ s

Neutrino decoupling

Primordial nucleosynthesis

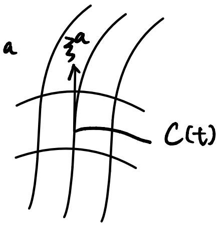
… …

Solve Einstein equation

Df. A spacetime (M, g_{ab}) is said to be stationary, if there exists a timelike killing vector field ξ^a

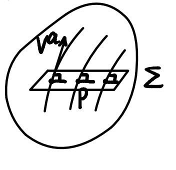
找到一个 ξ^a 的适配坐标系 $\xi^a = (\frac{\partial}{\partial t})^a$

(以t为一个坐标)



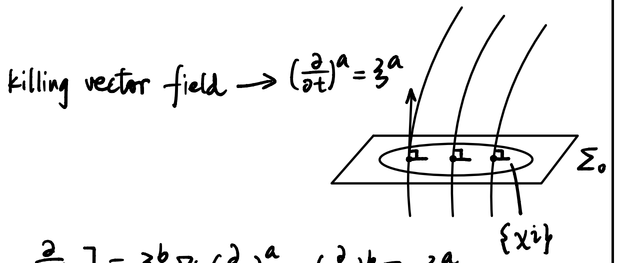
$$\frac{\partial g_{\mu\nu}}{\partial t} = (\mathcal{L}_\xi g)_{\mu\nu} = 0 \Rightarrow g_{\mu\nu}(x^i)$$

Df. hypersurface orthogonal vector field



$$p \in \Sigma, \ \Sigma \perp v^a$$

Df. A space time (M, g_{ab}) is said to be static, if there exists a hypersurface orthogonal killing vector field.



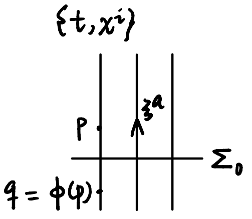
由 $0 = [\xi, \frac{\partial}{\partial x^i}] = \xi^b \nabla_b (\frac{\partial}{\partial x^i})^a - (\frac{\partial}{\partial x^i})^b \nabla_b \xi^a$

$$\xi^b \nabla_b (\xi_a (\frac{\partial}{\partial x^i})^a)$$
$$= \xi_a \xi^b \nabla_b (\frac{\partial}{\partial x^i})^a + (\frac{\partial}{\partial x^i})^a \xi^b \nabla_b \xi_a$$
$$= \xi_a (\frac{\partial}{\partial x^i})^b \nabla_b \xi^a + (\frac{\partial}{\partial x^i})^a \xi^b \nabla_b \xi_a$$
$$= 2 \xi^{(a} (\frac{\partial}{\partial x^i})^{b)} \nabla_b \xi_a = 2 \xi^a (\frac{\partial}{\partial x^i})^b \nabla_{(a} \xi_{b)}$$

对于 killing vector field $\nabla_{(b}\xi_{a)} = 0$ 所以恒有 $(\frac{\partial}{\partial x^i})^a \xi_a = 0$

$$ds^2 = g_{00}(x) dt^2 + g_{ij}(x) dx^i dx^j$$

对一个 static spacetime



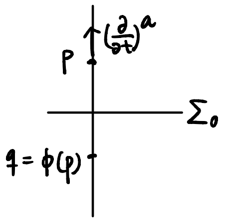
time reflection $\phi: M \to M$

$\forall p \in M, \phi(p)$ $\quad t(\phi(p)) := -t(p)$

$\qquad x^i(\phi(p)) := x^i(p)$

Claim: for static spacetime, ϕ is an isometry

Pf: $p = C(t_1)$, $q = \phi(p) = C(-t_1)$



$$\phi_*(\frac{\partial}{\partial t})^a|_p (f) = (\frac{\partial}{\partial t})^a|_q (\phi^* f)$$

令 $f = t$ $\quad \phi^* t = -t$

$$\phi_*(\frac{\partial}{\partial t})^a|_p (t) = (\frac{\partial}{\partial t})^a|_q (-t)$$

$$\Rightarrow \phi_*(\frac{\partial}{\partial t})^a (dt)_a = -(\frac{\partial}{\partial t})^a (dt)_a$$

而 $f = x^i$，又有 $\phi_*(\frac{\partial}{\partial t})^a (dx^i)_a = (\frac{\partial}{\partial t})^a (dx^i)_a = 0$

$$\Rightarrow \phi_*(\frac{\partial}{\partial t})^a|_p = -(\frac{\partial}{\partial t})^a|_q$$

同理 $\phi_*(\frac{\partial}{\partial x^i})^a|_p = (\frac{\partial}{\partial x^i})^a|_q$

$$(\phi^* g)_{00}|_p = [(\phi^* g)_{ab} (\frac{\partial}{\partial t})^a (\frac{\partial}{\partial t})^b]_p$$
$$= g_{ab}|_q \phi_*(\frac{\partial}{\partial t})^a \phi_*(\frac{\partial}{\partial t})^b = g_{00}|_q = g_{00}|_p$$

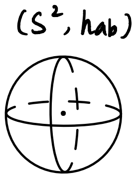
$$(\phi^* g)_{0i}|_p = [(\phi^* g)_{ab} (\frac{\partial}{\partial t})^a (\frac{\partial}{\partial x^i})^b]_p$$
$$= g_{ab}|_q \phi_*(\frac{\partial}{\partial t})^a \phi_*(\frac{\partial}{\partial x^i})^b = -g_{0i}|_q = 0 = g_{0i}|_p$$

$$(\phi^* g)_{ij}|_p = \cdots = g_{ij}|_q = g_{ij}|_p$$

$$(\mathcal{L}_{\frac{\partial}{\partial t}} g_{ab} = 0)$$

Spherically symmetric spacetimes

$$ds^2 = r^2(d\theta^2 + \sin^2\theta \, d\phi^2)$$



$$\xi_1^a = \left(\frac{\partial}{\partial\phi}\right)^a$$

$$\xi_2^a = \left(\frac{\partial}{\partial\theta}\right)^a \sin\phi + \left(\frac{\partial}{\partial\phi}\right)^a \cot\theta\cos\phi$$

$$\xi_3^a = \left(\frac{\partial}{\partial\theta}\right)^a \cos\phi - \left(\frac{\partial}{\partial\phi}\right)^a \cot\theta\sin\phi$$

isometry group — 3 dim $G_i \xleftrightarrow[\text{isomorphic}]{} SO(3)$

the orbit through p of isometry group G_i

Df. A spacetime is said to be spherically symmetric if its isometry group contains a subgroup isomorphic to $SO(3)$ and each of its orbits is an S^2.

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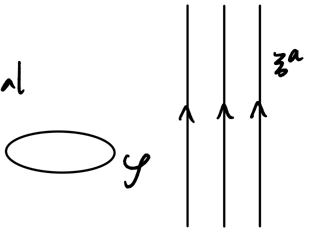
static spherically symmetric spacetime



Claim: 如果 hypersurface orthogonal killing vector field 是 unique

则 $\xi^a \perp \mathcal{S}$

Pf. $\phi \in G_3$, $\phi_* \xi^a$ 也是 hypersurface orthogonal killing vector field $\underset{\text{unique}}{\Rightarrow} \phi_*\xi^a = \xi^a$

所以 ξ^a 没有在 $\mathcal{S}$ 上的投影分量

$\Rightarrow \xi^a \perp \mathcal{S}$ □

$$ds^2 = \underset{\substack{\downarrow \\ -e^{2A(r)}}}{g_{00}(r)} dt^2 + \underset{\substack{\downarrow \\ e^{2B(r)}}}{g_{11}(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

The Schwarzschild vacuum solution

$$R_{ab} - \frac{1}{2}Rg_{ab} = 0 \Rightarrow R_{ab} = 0$$

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$g_{\mu\nu} = \begin{pmatrix} -e^{2A(r)} & & & \\ & e^{2B(r)} & & \\ & & r^2 & \\ & & & r^2\sin^2\theta \end{pmatrix} \quad \begin{array}{cccc} 0 & 1 & 2 & 3 \\ \{t, & r, & \theta, & \phi\} \end{array}$$

$$R_{abc}{}^d = -2\partial_{[a}\Gamma^d_{b]c} + 2\Gamma^e_{c[a}\Gamma^d_{b]e}$$

$$R_{ac} = R_{abc}{}^b$$

$$\Gamma^0_{00} = \frac{1}{2}(-e^{-2A})(0) = 0$$

$$\Gamma^0_{01} = \frac{1}{2}(-e^{-2A})(-\partial_r e^{2A}) = \partial_r A$$ ✓

$$\Gamma^0_{02} = \Gamma^0_{03} = 0$$

$$\Gamma^0_{11} = \frac{1}{2}(-e^{-2A})(-0) = 0 = \Gamma^0_{22} = \Gamma^0_{33}$$

$$\Gamma^1_{10} = \frac{1}{2}(e^{-2B})(0) = 0$$

$$\Gamma^1_{11} = \frac{1}{2}(e^{-2B})(\partial_r e^{2B}) = \partial_r B$$ ✓

$$\Gamma^1_{12} = \Gamma^1_{13} = 0$$

$$\Gamma^1_{00} = \frac{1}{2}(e^{-2B})(-\partial_r(-e^{2A})) = e^{2(A-B)}\partial_r A$$ ✓

$$\Gamma^1_{22} = \frac{1}{2}(e^{-2B})(-\partial_r(r^2)) = -re^{-2B}$$ ✓

$$\Gamma^1_{33} = \frac{1}{2}(e^{-2B})(-\partial_r(r^2\sin^2\theta)) = -r\sin^2\theta\, e^{-2B}$$ ✓

$$\Gamma^2_{00} = \Gamma^2_{11} = \Gamma^2_{22} = 0$$

$$\Gamma^2_{33} = \frac{1}{2}\left(\frac{1}{r^2}\right)(-\partial_\theta(r^2\sin^2\theta)) = -\sin\theta\cos\theta = -\frac{1}{2}\sin 2\theta$$ ✓

$$\Gamma^2_{20} = 0 \qquad \Gamma^2_{23} = 0$$

$$\Gamma^2_{21} = \frac{1}{2}\left(\frac{1}{r^2}\right)(\partial_r(r^2)) = \frac{1}{r}$$ ✓

$$\Gamma^3_{00} = \Gamma^3_{11} = \Gamma^3_{22} = \Gamma^3_{33} = 0$$

$$\Gamma^3_{30} = 0$$

$$\Gamma^3_{31} = \frac{1}{2}\left(\frac{1}{r^2\sin^2\theta}\right)(\partial_r(r^2\sin^2\theta)) = \frac{1}{r}$$ ✓

$$\Gamma^3_{32} = \frac{1}{2}\left(\frac{1}{r^2\sin^2\theta}\right)(\partial_\theta(r^2\sin^2\theta)) = \cot\theta$$ ✓

$$R_{\mu\nu\rho}{}^{\sigma} = -2\partial_{[\mu}\Gamma^{\sigma}_{\nu]\rho} + 2\Gamma^{\tau}_{\rho[\mu}\Gamma^{\sigma}_{\nu]\tau}$$

$$\underline{R_{010}{}^{1}} = -2\partial_{[0}\Gamma^{1}_{1]0} + 2\Gamma^{\tau}_{0[0}\Gamma^{1}_{1]\tau}$$

$$= e^{2(A-B)}\left[\partial_r A(\partial_r A - \partial_r B) + \partial_r^2 A\right] \quad ✓$$

$$R_{010}{}^{2} = -2\partial_{[0}\Gamma^{2}_{1]0} + 2\Gamma^{\tau}_{0[0}\Gamma^{2}_{1]\tau} = 0$$

$$R_{010}{}^{3} = -2\partial_{[0}\Gamma^{3}_{1]0} + 2\Gamma^{\tau}_{0[0}\Gamma^{3}_{1]\tau} = 0$$

$$R_{011}{}^{2} = -2\partial_{[0}\Gamma^{2}_{1]1} + 2\Gamma^{\tau}_{1[0}\Gamma^{2}_{1]\tau} = 0$$

$$R_{011}{}^{3} = -2\partial_{[0}\Gamma^{3}_{1]1} + 2\Gamma^{\tau}_{1[0}\Gamma^{3}_{1]\tau} = 0$$

$$R_{012}{}^{3} = 0$$

$$\underline{R_{020}{}^{2}} = -2\partial_{[0}\Gamma^{2}_{2]0} + 2\Gamma^{\tau}_{0[0}\Gamma^{2}_{2]\tau} = \frac{1}{r}e^{2(A-B)}\partial_r A \quad ✓$$

$$R_{020}{}^{3} = -2\partial_{[0}\Gamma^{3}_{2]0} + 2\Gamma^{\tau}_{0[0}\Gamma^{3}_{2]\tau} = 0$$

$$R_{021}{}^{2} = -2\partial_{[0}\Gamma^{2}_{2]1} + 2\Gamma^{\tau}_{1[0}\Gamma^{2}_{2]\tau} = 0$$

$$R_{021}{}^{3} = 0$$

$$R_{022}{}^{3} = -2\partial_{[0}\Gamma^{3}_{2]2} + 2\Gamma^{\tau}_{2[0}\Gamma^{3}_{2]\tau} = 0$$

$$\underline{R_{030}{}^{3}} = -2\partial_{[0}\Gamma^{3}_{3]0} + 2\Gamma^{\tau}_{0[0}\Gamma^{3}_{3]\tau} = \frac{1}{r}e^{2(A-B)}\partial_r A \quad ✓$$

$$R_{031}{}^{2} = 0$$

$$R_{031}{}^{3} = -2\partial_{[0}\Gamma^{3}_{3]1} + 2\Gamma^{\tau}_{1[0}\Gamma^{3}_{3]\tau} = 0$$

$$R_{032}{}^{3} = -2\partial_{[0}\Gamma^{3}_{3]2} + 2\Gamma^{\tau}_{2[0}\Gamma^{3}_{3]\tau} = 0$$

$$\underline{R_{121}{}^{2}} = -2\partial_{[1}\Gamma^{2}_{2]1} + 2\Gamma^{\tau}_{1[1}\Gamma^{2}_{2]\tau} = \frac{1}{r}\partial_r B \quad ✓$$

$$R_{121}{}^{3} = -2\partial_{[1}\Gamma^{3}_{2]1} + 2\Gamma^{\tau}_{1[1}\Gamma^{3}_{2]\tau} = 0$$

$$R_{122}{}^{3} = -2\partial_{[1}\Gamma^{3}_{2]2} + 2\Gamma^{\tau}_{2[1}\Gamma^{3}_{2]\tau} = 0$$

$$\underline{R_{131}{}^{3}} = -2\partial_{[1}\Gamma^{3}_{3]1} + 2\Gamma^{\tau}_{1[1}\Gamma^{3}_{3]\tau} = \frac{1}{r}\partial_r B \boxed{-\frac{1}{r^2}}$$

$$R_{132}{}^{3} = -2\partial_{[1}\Gamma^{3}_{3]1} + 2\Gamma^{\tau}_{2[1}\Gamma^{3}_{3]\tau} = 0$$

$$\underline{R_{232}{}^{3}} = -2\partial_{[2}\Gamma^{3}_{3]2} + 2\Gamma^{\tau}_{2[2}\Gamma^{3}_{3]\tau} = 1 - e^{-2B} \quad ✓$$

$$R_{00} = e^{2(A-B)}\left[\partial_r A(\partial_r A - \partial_r B) + \partial_r^2 A + \frac{2}{r}\partial_r A\right]$$

$$R_{01} = R_{0\mu 1}{}^{\mu} = 0$$

$$R_{02} = R_{0\mu 2}{}^{\mu} = 0$$

$$R_{03} = R_{0\mu 3}{}^{\mu} = 0$$

$$R_{11} = R_{1\mu 1}{}^{\mu} = -\left[\partial_r A(\partial_r A - \partial_r B) + \partial_r^2 A\right] + \frac{2}{r}\partial_r B \boxed{-\frac{1}{r^2}}$$

$$R_{12} = R_{1\mu 2}{}^{\mu} = 0$$

$$R_{13} = 0$$

$$R_{22} = R_{2\mu 2}{}^{\mu} = re^{-2B}(\partial_r B - \partial_r A) + 1 - e^{-2B}$$

$$R_{23} = R_{2\mu 3}{}^{\mu} = 0$$

$$R_{33} = R_{3\mu 3}{}^{\mu} = \sin^2\theta\left[re^{-2B}(\partial_r B - \partial_r A) + 1 - \boxed{2}e^{-2B}\right]$$

于是 Einstein equation 化为

$$\begin{cases} e^{2(A-B)}\left[\partial_r A(\partial_r A - \partial_r B) + \partial_r^2 A + \frac{2}{r}\partial_r A\right] = 0 & ① \\ -\left[\partial_r A(\partial_r A - \partial_r B) + \partial_r^2 A\right] + \frac{2}{r}\partial_r B = 0 & ② \\ re^{-2B}(\partial_r B - \partial_r A) + 1 - e^{-2B} = 0 & ③ \\ \sin^2\theta\left[re^{-2B}(\partial_r B - \partial_r A) + 1 - e^{-2B}\right] = 0 & ④ \end{cases}$$

解得： $e^{2A} = 1 + \frac{c}{r}$ （注意，由于④，这里没有整体的乘子系数）

$$e^{2B} = \frac{1}{1 + \frac{c}{r}}$$

$$\therefore ds^2 = -\left(1 + \frac{c}{r}\right)dt^2 + \frac{1}{1 + \frac{c}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

根据牛顿近似 $c = -2M$

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \frac{1}{1 - \frac{2M}{r}}dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2)$$

Claim: for static observer $A^a = \nabla^a \ln \chi$, $\chi = \sqrt{-\xi^a \xi_a}$

Pf.
$$U^a U_a = -1 \Rightarrow U^a = \frac{\xi^a}{\chi}$$

$$A^a = \frac{\xi^b}{\chi} \nabla_b \frac{\xi^a}{\chi}$$

又 $\nabla_{(a} \xi_{b)} = 0$, 代入

$$A^a = \frac{\xi^b}{\chi^2} \nabla_b \xi^a + \frac{\xi^b \xi_a}{\chi} \nabla_b \frac{1}{\chi}$$

$$= -\frac{\xi^b}{\chi^2} \nabla^a \xi_b + \underset{\substack{\downarrow \\ 0}}{\frac{\xi^b \xi_a \xi^c}{\chi^4} \nabla_b \xi_c}$$

$$= \frac{1}{2} \frac{1}{\chi^2} \nabla^a \chi^2$$

$$= \frac{1}{2} \nabla^a \ln \chi^2 = \nabla^a \ln \chi \qquad \square$$

Birkhoff's theorem

The Reissner – Nordstrom solution

electrovacuum：只有电磁场的真空

Einstein-Maxwell Equation

$$\begin{cases} R_{ab} - \frac{1}{2} g_{ab} R = 8\pi T_{ab} \\ T_{ab} = \frac{1}{4\pi}\left(F_{ac} F_b{}^c - \frac{1}{4} g_{ab} F_{cd} F^{cd} \right) \\ \nabla^a F_{ab} = 0 \\ \nabla_{[a} F_{bc]} = 0 \end{cases}$$

null electromagnetic field

$$\Sigma_{ab} = F_{ab} + i\, {}^*F_{ab}$$

$$\Sigma_{ab} \Sigma^{ab} = 2\left(F_{ab} F^{ab} + i\, F_{ab} {}^*F^{ab} \right) = 0$$

则 F_{ab} 是 null electromagnetic field

$$F_{ab} F^{ab} = 2(B^2 - E^2)$$

$$F_{ab} {}^*F^{ab} = 4 \vec{E} \cdot \vec{B}$$

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2 (d\theta^2 + \sin^2\theta\, d\phi^2)$$

$$\{\overset{0}{t}, \overset{1}{r}, \overset{2}{\theta}, \overset{3}{\phi}\}$$

$F_{ab} = 2\partial_{[a} A_{b]}$ 由 sph. sym. $A_2 = 0, A_3 = 0$

通过 gauge transformation $\tilde{A}_a = A_a + \nabla_a \chi$

$$\tilde{A}_1 = A_1 + \partial_r \chi$$

最后可得 $A_0(r)$ 其余 $A_1 . A_2 . A_3$ 都为 0

$F_{01} = -\partial_r A_0 \quad (= -F_{10})$ 其它 $F_{\mu\nu} = 0$

$$0 = F^{\mu\nu}{}_{;\mu} = F^{\mu\nu}{}_{,\mu} + \Gamma^{\mu}_{\rho\mu} F^{\rho\nu} + \underset{\to 0}{\Gamma^{\nu}_{\rho\mu} F^{\mu\rho}}$$

$$= F^{\mu\nu}{}_{,\mu} + (\ln\sqrt{-g})_{,\rho} F^{\rho\nu}$$

$$= \frac{1}{\sqrt{-g}} (\sqrt{-g} F^{\mu\nu})_{,\mu}$$

$$g_{\mu\nu} = \begin{pmatrix} -e^{2\alpha(r)} & & & \\ & e^{2\beta(r)} & & \\ & & r^2 & \\ & & & r^2 \sin^2\theta \end{pmatrix}$$

$$F^{01} = -e^{-2\alpha} e^{-2\beta} F_{01}$$

$$= -e^{-2(\alpha+\beta)} \partial_r A_0$$

$$\Rightarrow \begin{cases} 0 = \partial_r (\sqrt{-g} F^{10}) = \partial_r (e^{-(\alpha+\beta)} r^2 \sin\theta\, \partial_r A_0) \\ 0 = \partial_t (\sqrt{-g} F^{01}) \text{，(恒成立)} \end{cases}$$

$$\Rightarrow 0 = \partial_r \left(\overset{\curvearrowright \text{Constant}}{e^{-(\alpha+\beta)} r^2 \partial_r A_0} \right)$$

$$\Rightarrow \partial_r A_0 = \frac{Q}{r^2} e^{(\alpha+\beta)}$$

$$F_{10} = \frac{Q}{r^2} e^{(\alpha+\beta)}$$

Claim: for electromagnetic field $T = T^a{}_a = 0$

Pf: $T = \frac{1}{4\pi}\left(F_{ab} F^{ab} - \frac{1}{4} g^a{}_a F_{bc} F^{bc} \right) = 0$

vac.: $R_{ab} = 0$

electrovac.: $R = 0 \Leftrightarrow R_{ab} = 8\pi T_{ab}$

$$T_{\mu\nu} = \frac{1}{4\pi}\left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right)$$

$$F^{\rho\sigma} F_{\rho\sigma} = 2 F_{01}{}^2 g^{00} g^{11} = 2 \frac{Q^2}{r4} e^{2(\alpha+\beta)} (-e^{-2(\alpha+\beta)}) = -2 \frac{Q^2}{r4}$$

$$F_{1\rho} F_1{}^\rho = \frac{Q^2}{r4} e^{2(\alpha+\beta)} g^{00} = -e^{2\beta} \frac{Q^2}{r4}$$

$$F_{0\rho} F_0{}^\rho = \frac{Q^2}{r4} e^{2(\alpha+\beta)} g^{11} = e^{2\alpha} \frac{Q^2}{r4}$$

当 $\mu\nu \neq 00$ or 11 时 $\quad T_{\mu\nu} = \frac{1}{8\pi} \frac{Q^2}{r4} g_{\mu\nu}$

$$T_{00} = \frac{1}{4\pi}\left(e^{2\alpha} \frac{Q^2}{r4} + \frac{1}{4} e^{2\alpha} \left(-2 \frac{Q^2}{r4}\right) \right)$$

$$= \frac{1}{8\pi} e^{2\alpha} \frac{Q^2}{r4}$$

$$T_{11} = \frac{1}{4\pi}\left(-e^{2\beta} \frac{Q^2}{r4} - \frac{1}{4} e^{2\beta} \left(-2 \frac{Q^2}{r4}\right) \right)$$

$$= -\frac{1}{8\pi} e^{2\beta} \frac{Q^2}{r4}$$

$$T_{22} = \frac{1}{8\pi} \frac{Q^2}{r^2}$$

$$T_{33} = \frac{1}{8\pi} \frac{Q^2}{r^2} \sin^2\theta$$

代入 Einstein's equations

$$\begin{cases} e^{2(\alpha-\beta)}\left[\partial r\alpha(\partial r\alpha-\partial r\beta)+\partial r^2\alpha+\frac{2}{r}\partial r\alpha\right]=e^{2\alpha}\frac{Q^2}{r4} & ① \\ -[\partial r\alpha(\partial r\alpha-\partial r\beta)+\partial r^2\alpha]+\frac{2}{r}\partial r\beta=-e^{2\beta}\frac{Q^2}{r4} & ② \\ re^{-2\beta}(\partial r\beta-\partial r\alpha)+1-e^{-2\beta}=\frac{Q^2}{r^2} & ③ \\ \sin^2\theta\left[re^{-2\beta}(\partial r\beta-\partial r\alpha)+1-e^{-2\beta}\right]=\frac{Q^2}{r^2}\sin^2\theta & ④ \end{cases}$$

①、② ⇒

$$\partial r\alpha(\partial r\alpha-\partial r\beta)+\partial_r^2\alpha+\frac{2}{r}\partial r\alpha$$

$$=\partial r\alpha(\partial r\alpha-\partial r\beta)+\partial_r^2\alpha-\frac{2}{r}\partial r\beta=e^{2\beta}\frac{Q^2}{r4}$$

$$\Rightarrow\begin{cases}\alpha+\beta=\text{Const}\quad \text{i.e. } e^{2\alpha}=\frac{k}{e^{2\beta}} & ⑤\\ 2(\partial r\alpha)^2+\partial_r^2\alpha+\frac{2}{r}\partial r\alpha=\frac{k}{e^{2\alpha}}\frac{Q^2}{r4} & ⑥\end{cases}$$

⑤ 代入 ③ (④)

$$-2r\frac{e^{2\alpha}}{k}\partial r\alpha+1-\frac{e^{2\alpha}}{k}=\frac{Q^2}{r^2}$$

$$\Rightarrow\ \partial r\left(r\frac{e^{2\alpha}}{k}\right)=1-\frac{Q^2}{r^2}$$

$$e^{2\alpha}=k\left(1+\frac{Q^2}{r^2}+\frac{C}{r}\right)\qquad e^{2\beta}=\left(1+\frac{Q^2}{r^2}+\frac{C}{r}\right)^{-1}$$

The Reissner–Nordstrom solution:

$$ds^2=-\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)dt^2+\frac{1}{1-\frac{2M}{r}+\frac{Q^2}{r^2}}dr^2+r^2(d\theta^2+\sin^2\theta d\phi^2)$$

$$F_{ab}=-\frac{Q}{r^2}(dt)_a\wedge(dr)_b$$

$$E_a=F_{ab}Z^b\qquad B_a=-{}^*F_{ab}Z^b=-\frac{1}{2}F^{cd}\varepsilon_{cdab}\ Z^b$$

而对于 static observer $Z^b=\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-\frac{1}{2}}\left(\frac{\partial}{\partial t}\right)^b$

$$E_a=\frac{Q}{r^2}\left(1-\frac{2M}{r}+\frac{Q^2}{r^2}\right)^{-\frac{1}{2}}(dr)_a\qquad B_a=0$$

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axisymmetric & cylindrically symmetric & plane symmetric

plane symmetric + vaccum + static

$$ds^2=\frac{1}{\sqrt{1+kZ}}(-dT^2+dZ^2)+(1+kZ)(dX^2+dY^2)$$

如果不要求 static，另有一解

$$ds^2=-\frac{1}{\sqrt{1+kZ}}(-dT^2+dZ^2)+(1+kZ)(dX^2+dY^2)$$

换元 $t=k^{-\frac{1}{3}}T$，$z=k^{-\frac{4}{3}}Z$，$x=k^{\frac{2}{3}}X$，$y=k^{\frac{2}{3}}Y$

$$ds^2=z^{-\frac{1}{2}}(-dt^2+dz^2)+z(dx^2+dy^2)$$

$$ds^2=-z^{-\frac{1}{2}}(-dt^2+dz^2)+z(dx^2+dy^2)$$